On Uncertainty Principles of the Fourier transform

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This thesis surveys the vast landscape of uncertainty principles of the Fourier transform. The research of these uncertainty principles began in the mid 1920's following a seminal lecture by Wiener, where he first gave the remark that condenses the idea of uncertainty principles: "A function and its Fourier transform cannot be simultaneously arbitrarily small".

In this thesis we examine some of the most remarkable classical results where different interpretations of smallness is applied. Also more modern results and links to active fields of research are presented. We make great effort to give an extensive list of references to build a good broad understanding of the subject matter.

Chapter 2 gives the reader a sufficient basic theory to understand the contents of this thesis. First we talk about Hilbert spaces and the Fourier transform. Since they are very central concepts in this thesis, we try to make sure that the reader can get a proper understanding of these subjects from our description of them. Next, we study Sobolev spaces and especially the regularity properties of Sobolev functions. After briefly looking at tempered distributions we conclude the chapter by presenting the most famous of all uncertainty principles, Heisenberg's uncertainty principle.

In chapter 3 we examine how the rate of decay of a function affects the rate of decay of its Fourier transform. This is the most historically significant form of the uncertainty principle and therefore many classical results are presented, most importantly the ones by Hardy and Beurling. In 2012 Hedenmalm gave a beautiful new proof to the result of Beurling. We present the proof after which we briefly talk about the Gaussian function and how it acts as the extremal case of many of the mentioned results.

In chapter 4 we study how the support of a function affects the support and regularity of its Fourier transform. The magnificent result by Benedicks and the results following it work as the focal point of this chapter but we also briefly talk about the Gap problem, a classical problem with recent developments.

Chapter 5 links density based uncertainty principle to Fourier quasicrystals, a very active field of research. We follow the unpublished work of Kulikov-Nazarov-Sodin where first an uncertainty principle is given, after which a formula for generating Fourier quasicrystals, where a density condition from the uncertainty principle is used, is proved. We end by comparing this formula to other recent formulas generating quasicrystals.

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Chapter 1

Introduction

The Fourier transform is one of the most important integral transforms and has a far reaching range of utility. It offers a way to ease the analysis of possibly cumbersome operations, such as derivation and convolution. Also, for a signal it gives information on frequencies present and their proportions. These properties already make it immensely useful for example in the study of partial differential equations and signal processing. With a plethora of other very useful traits, the Fourier transform is also often used in a wide variety of other mathematical disciplines.

In this thesis we will study one particularly interesting property of the Fourier transform, the *uncertainty principle.* In a broad sense the meaning of the uncertainty principle can be condensed to the following sentence:

> A function[1] and its Fourier transform cannot be simultaneously arbitrarily small.

The idea dates back to 1925 when Wiener gave an influential lecture in Göttingen on Harmonic analysis. Interestingly enough, the man most often associated with the uncertainty principle, Heisenberg, was at the time working on the foundations of quantum mechanics in Göttingen. Two years later Heisenberg published the seminal paper [18] that introduced the idea of the uncertainty principle in the context of quantum mechanics.

The word "small" appearing in the characterization of the uncertainty principle is obviously ambiguous in mathematics, but a huge variety of different interpretations have been formulated into results after Wiener's lecture. In this thesis we will study a few of the most fundamental classes of uncertainty principles. First, in chapter 3 we study the interplay between the rate of decay of the function and its Fourier transform. In chapter 4 we investigate uncertainty principles that focus on the support of a function. Finally in

[1] More precisely a L^2-normalized function.

chapter 5, we present a very recent, still unpublished, density based uncertainty principle by Kulikov-Nazarov-Sodin, and we link density based uncertainty principles to Fourier quasicrystals, currently a very active field of research. Chapter 2 offers the needed basic theory for a reader, although we assume that the reader has basic knowledge of L^p-spaces and complex analysis.

As the subject matter is so broad we do not even dream of providing a comprehensive survey. From the list of references, an eager reader should first concentrate on the monograph on the subject by Havin and Jöricke [16].

Few remarks on the notation

- For a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$, we denote $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and $D^\alpha = (\partial_{x_1})^{\alpha_1}(\partial_{x_2})^{\alpha_2} \ldots (\partial_{x_d})^{\alpha_d}$, where ∂_x is the partial derivative with respect to x.
- If $(X, \|\cdot\|)$ is a normed space, we denote $B = \{x \in X : \|x\| < 1\}$.
- If there exists a $M > 0$, such that there is $x_0 > 0$ for which $|f(x)| \leq Mg(x)$ for all $x \leq x_0$, we denote $f(x) = \mathcal{O}(g(x))$.
- If $a \leq Cb$ for some constant $C > 0$, we denote $a \lesssim b$.
- If Ω is a set, then χ_Ω is the characteristic function of Ω.

Chapter 2

Prerequisites

Before we jump to the uncertainty principles, some theory needed to understand the results is given in this chapter. Some of the proofs are omitted, but we give ample selection of literature where a reader can find them.

2.1 Hilbert spaces

We start with a fundamental concept of functional analysis, the Hilbert space. In this chapter we utilize the great books by Rudin [43, 44] to build most of the base theory. This section is no exception.

Definition 2.1. Let H be a complex vector space. We say that H is an *inner product space* if we can define a function $\langle \cdot, \cdot \rangle \colon H \times H \to \mathbb{C}, (x, y) \mapsto \langle x, y \rangle$ for which the following conditions hold:

(I1) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in H$.

(I2) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, for all $x, y, z \in H$.

(I3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in H$ and $\alpha \in \mathbb{C}$.

(I4) $\langle x, x \rangle \geq 0$ for all $x \in H$.

(I5) $\langle x, x \rangle = 0$ only if $x = 0$.

If these conditions hold, we call $\langle \cdot, \cdot \rangle$ the *inner product*.

We show some straightforward properties that rise from the definition:

- The axioms (I2) and (I3) imply linearity of the inner product with respect to the first variable.
- By (I3) we have $\langle 0, y\rangle = \langle 2\cdot 0, y\rangle = 2\langle 0, y\rangle$ and therefore $\langle 0, y\rangle = 0$ for all $y \in H$.
- From the axioms (I1) and (I2) we get $\langle x, y+z\rangle = \langle x, y\rangle + \langle y, z\rangle$, for all $x, y, z \in H$.
- The axioms (I1) and (I3) give us $\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle} = \overline{\alpha\langle y, x\rangle} = \overline{\alpha}\langle x, y\rangle$.
- The last two properties together imply that the inner product is *antilinear* respect to the second variable.

We denote $\|x\| = \sqrt{\langle x, x\rangle}$ and aim to show that $\|\cdot\|$ is a norm. As a byproduct we acquire the incredibly handy Cauchy-Schwarz inequality.

Theorem 2.2 (Cauchy-Schwarz inequality)**.** *For all $x, y \in H$ we have*

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \tag{2.1}$$

where we have equality only if x and y are linearly dependent.

Proof. If $x = 0$ or $y = 0$, the claim is trivially true. We therefore assume that $x \neq 0 \neq y$. For any $\lambda \in \mathbb{C}$ we use the properties of the inner product to obtain

$$\begin{aligned} 0 &\leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle = \langle x, x - \lambda y\rangle + \langle -\lambda y, x - \lambda y\rangle \\ &= \|x\|^2 - \overline{\lambda}\langle x, y\rangle - \lambda\overline{\langle x, y\rangle} + |\lambda|^2\|y\|^2 \end{aligned}$$

and by choosing $\lambda = \frac{\langle x,y\rangle}{\|y\|^2}$ we get

$$0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} - \frac{|\langle x, y\rangle|^2}{\|y\|^2} + \frac{|\langle x, y\rangle|^2}{\|y\|^4}\|y\|^2.$$

This can be cleaned up to

$$|\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2$$

which proves (2.1).

To prove condition for equality we first assume $|\langle x, y\rangle| = \|x\|\|y\|$ and by utilizing the previous deduction in reverse we end up with

$$\left\| x - \frac{\langle x, y\rangle}{\|y\|^2} y \right\|^2 = 0,$$

which gives us $x = \frac{\langle x,y\rangle}{\|y\|^2} y$.

Finally, we assume that $x = \lambda y$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ whence

$$|\langle x, y\rangle| = |\langle \lambda y, y\rangle| = |\lambda|\|y\|^2 = \|\lambda y\|\|y\| = \|x\|\|y\|,$$

which concludes the proof. $\square$

With this inequality it is very straightforward to conclude that $\|\cdot\|$ really is a norm.

Theorem 2.3. *Let H be an inner product space. Then the inner product of H induces a norm $\|x\| = \sqrt{\langle x, x\rangle}$.*

Proof. We simply check that $\|\cdot\|$ fulfills the norm axioms.

(N1) By using the Cauchy-Schwarz inequality we get for all $x, y \in H$

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.\end{aligned}$$

(N2) Let $x \in H$ and $\alpha \in \mathbb{C}$. By simple calculation

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\overline{\alpha}\langle x, x\rangle} = |\alpha|\|x\|.$$

(N3) This follows immediately from (I5).

$\square$

For later use it is useful to note that the inner product also works very nicely with regards to continuity

Theorem 2.4. *Let y be a fixed point in H. Now the mappings*

$$x \mapsto \langle x, y\rangle, \quad x \mapsto \langle y, x\rangle, \quad x \mapsto \|x\|$$

are all continuous in H.

Proof. Let $y \in H$ be fixed and $\varepsilon > 0$. We choose $\delta = \varepsilon/\|y\|$, take $x_1, x_2 \in H$ such that $\|x_1 - x_2\| < \delta$ and use Cauchy-Schwarz to get

$$|\langle x_1, y\rangle - \langle x_2, y\rangle| = \|\langle x_1 - x_2, y\rangle\| \leq \|x_1 - x_2\|\|y\| < \epsilon.$$

This gives us uniform continuity of $x \mapsto \langle x, y\rangle$. We get uniform continuity of $x \mapsto \langle y, x\rangle$ similarly.

The uniform continuity of $x \mapsto \|x\|$ follows from the fact that every norm is a uniformly continuous mapping. $\square$

Next, we will recall the definition of a Hilbert space. After that we will showcase some of its properties (especially the ones that we need later on) that make it such a ubiquitous structure. One thing that makes Hilbert spaces interesting is the fact that they are a generalization of Eucilidian spaces, which preserves important properties when moving to an infinite-dimensional situation.

Definition 2.5. Let H be an inner product space. If H is also complete, we call it a *Hilbert space.*

Due to the existence of inner product Hilbert spaces inherit the concept of orthogonality. The element $x \in H$ is said to be *orthogonal* to $y \in H$ if $\langle x, y \rangle = 0$. In this case we denote $x \perp y$. Next, let M be a subspace of H. Then the set $M^\perp$ of elements $x \in H$ that are orthogonal to all elements of $y \in M$ is called the *orthogonal complement of M.*

We show that the orthogonal complement $M^\perp$ is a closed set. First of all the set $\{y\}^\perp$ is closed as a preimage of $\{0\}$ under the continuous mapping $x \mapsto \langle x, y \rangle$. By definition

$$M^\perp = \bigcap_{y \in M} \{y\}^\perp$$

so as an intersection of closed sets the orthogonal complement is closed. Also, if $x, y \in M^\perp$ and a, b are scalars, then for every $z \in M$

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle = 0$$

giving us that $ax + by \in M^\perp$. Therefore $M^\perp$ is a closed subspace of H.

Hilbert spaces also inherit from the inner product generalized versions of some of the important geometrical notions of the Euclidian spaces.

Theorem 2.6 (Pythagoras)**.** *Let H be an inner product space. If $x_1, x_2, \dots x_n \in H$ are such that x_i is orthogonal to x_j for all $i \neq j$ then*

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2.$$

Theorem 2.7 (Parallelogram law)**.** *Let H be an inner product space. Now*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in H$.

Next we move towards defining the orthogonal projection.

Theorem 2.8. *Let H be a Hilbert space and $M \subset H$ a nonempty, closed and convex set. Then there exists a unique $x_0 \in M$ such that $\|x_0\| \leq \|x\|$ for all $x \in H$.*

Proof. Let $x, y \in H$. We use the parallelogram law on vectors $\frac{1}{2}x$ and $\frac{1}{2}y$ to get

$$\left\|\frac{x+y}{2}\right\|^2 + \frac{1}{4}\|x-y\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2.$$

From the convexity of M we know that $\frac{x+y}{2}$ is in M. If we denote $\delta = \inf\{\|x\| : x \in M\}$ then

$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\|\frac{x+y}{2}\right\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta. \tag{2.2}$$

If we have two norm minimizing elements x_0 and y_0, the previous inequality implies that $x = y$. We have therefore proved uniqueness.

For the existence, we first note that there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset M$ such that $\|y_n\| \to \delta$ when $n \to \infty$. Setting $x = y_n$ and $y = y_m$ in (2.2) we get that

$$\|y_n - y_m\|^2 \leq 2\|y_n\|^2 + 2\|y_m\|^2 - 4\delta \to 0$$

as $n \to \infty$, making $(y_n)_{n\in\mathbb{N}}$ a Cauchy sequence in M. Since M is a closed subset of a complete space, it is also complete and thus there exists $y \in M$ such that $y_n \xrightarrow{n\to\infty} y$. Since the norm mapping $x \mapsto \|x\|$ is continuous, we have

$$\|y\| = \lim_{n\to\infty} \|y_n\| = \delta$$

which proves the existence of the norm minimizing element. □

For any closed subspace M, we will use the minimizing element to acquire a unique decomposition $x = a + b$ for any $x \in H$, such that $a \in M$ and $b \in M^\perp$.

Theorem 2.9. *Let H be a Hilbert space and M a closed subspace of H. There exists linear continuous mappings $P\colon H \to M$ and $Q\colon H \to M^\perp$ such that $x = Px + Qx$ for any $x \in H$.*

Proof. Let $x \in M$. By assumption $x + M := \{x + y : y \in M\}$ is closed and by simple calculation also convex. We set Qx as the norm minimizing element of $x + M$. The existence and uniqueness comes from the previous theorem. Then we define $Px = x - Qx$. Since $Qx \in x + M$, we have $Px \in M$.

Next we want to show that $Qx \in M^\perp$, in other words $\langle Qx, y\rangle = 0$ for all $y \in M$. Since $\left\langle Qx, \frac{y}{\|y\|}\right\rangle = \frac{1}{\|y\|}\langle Qx, y\rangle$, it is sufficient to show this for all $y \in M$ such that $\|y\| = 1$. We know that $Qx + \alpha y \in x + M$ for all scalars α and thus can utilize the norm minimizing property of Qx to say

$$\langle Qx, Qx\rangle = \|Qx\|^2 \leq \|Qx - \alpha y\|^2 = \langle Qx - \alpha y, Qx - \alpha y\rangle.$$

This is equivalent to

$$0 \leq -\overline{\alpha}\langle Qx, y\rangle - \alpha\langle y, Qx\rangle + |\alpha|^2$$

and by choosing $\alpha = \langle Qx, y\rangle$ this simplifies to $0 \leq -|\langle Qx, y\rangle|^2$. From this we can conclude $\langle Qx, y\rangle$ and therefore $Qx \in M^{\perp}$.

Next for the uniqueness we assume $x = x_1 + x_2$ is also a decomposition where $x_1 \in M$ and $x_2 \in M^{\perp}$. Now $x_1 + x_2 = Px + Qx$ or equivalently $x_1 - Px = Qx - x_2$. We combine $x_1 - Px \in M$ and $Qx - x_2 \in M^{\perp}$ to the knowledge that in a Hilbert space the origin is the only element orthogonal to itself to obtain $x_1 - Px = 0 = Qx - x_2$. Thus $x_1 = Px$ and $x_2 = Qx$ which proves the uniqueness of the decomposition.

Let $x, y \in H$ and $a, b \in \mathbb{C}$. By definition we get the compositions

$$\begin{aligned} ax + by &= P(ax + by) + Q(ax + by) \\ ax + by &= aPx + aQx + bPy + bQy. \end{aligned}$$

Since $P(ax + by) - aPx - bPy \in M$ and $-Q(ax + by) + aQx + bQy \in M^{\perp}$, the previous compositions give us

$$P(ax + by) - aPx - bPy = -Q(ax + by) + aQx + bQy \in M \cap M^{\perp} = \{0\},$$

which implies $P(ax + by) = aPx + bPy$ and $Q(ax + by) = aQx + bQy$. This verifies the linearity of P and Q.

The continuity follows from noticing that for every $x \in H$ we have by Pythagoras that

$$\|x\|^2 = \|Px + Qx\|^2 = \|Px\|^2 + \|Qx\|^2 \geq \|Px\|^2$$

which implies $\|Px\| \leq \|x\|$ and therefore continuity of P. An identical argument concludes the continuity of Q. $\square$

We call the mapping P from the previous theorem the *orthogonal projection* of H to M. By how it was defined in the theorem we immediately get that for any $x \in H$, the orthogonal projection Px is the closest element of M to the element x. The orthogonal projection helps us show one very powerful property of Hilbert spaces: every linear continuous functional can be represented with the inner product.

Theorem 2.10 (Riesz representation theorem)**.** *Let L be a continuous linear functional on a Hilbert space H. There exists a unique $y \in H$ such that*

$$Lx = \langle x, y\rangle \tag{2.3}$$

for all $x \in H$.

Proof. If $Lx = 0$ for all $x \in H$, then we can choose $y = 0$. Let $Lx \neq 0$ for some $x \in H$. We define the nonempty set $M = \{x \in H : Lx = 0\}$. As a kernel of a continuous linear map it is a closed subspace. If x_0 is the element of H such that $x_0 \notin M$, we know from the previous theorem that $x_0 = Px_0 + Qx_0$ where $Qx_0 \neq 0$ and $Qx_0 \in M^\perp$. By scaling Qx_0 we find an element $z \in M^\perp$ such that $\|z\| = 1$. Using z we define for given x

$$u = (Lx)z - (Lz)x$$

and immediately note that by linearity of L

$$Lu = (Lx)(Lz) - (Lz)(Lx) = 0,$$

thus giving us $u \in M$. Since $z \in M^\perp$ we have

$$0 = \langle u, z\rangle = \langle (Lx)z - (Lz)x, z\rangle$$

which is equivalent to $\langle (Lx)z, z\rangle = \langle (Lz)x, z\rangle$. Finally

$$Lx = (Lx)\|z\|^2 = \langle (Lx)z, z\rangle = \langle (Lz)x, z\rangle = \langle x, \overline{(Lz)}z\rangle$$

and thus (2.3) holds with the choice $y = \overline{(Lz)}z$.

For uniqueness we assume that (2.3) also holds for $y' \in H$. Then $\langle x, y\rangle = Lx = \langle x, y'\rangle$ holds for all $x \in H$. We choose $x = y - y'$ and get $0 = \langle y - y', y - y'\rangle = \|y - y'\|^2$, which implies $y = y'$. $\square$

In the final chapter we shall need a certain reproducing property of a special kind of Hilbert space.

Definition 2.11. Let H be a Hilbert space of complex valued functions on a set X. If for all $x \in X$ the linear evaluation functional $L_x : H \to \mathbb{C}$ defined by $L_x : f \mapsto f(x)$ is continuous, we call H a *reproducing kernel Hilbert space.*

The Riesz representation theorem implies that if H is a reproducing kernel Hilbert space of functions on X, then for every $x \in X$ there exists a $k_x \in H$ such that $f(x) = \langle f, k_x\rangle$ for all $f \in H$.

We finally give the definition of orthonormal sequences, but only to present one handy theorem we need later on.

Definition 2.12. Let A be an index set and $(e_n)_{n\in A}$ a sequence in a Hilbert space H. If $\|e_n\| = 1$ for all $n \in A$ and $e_i \perp e_j$ when $i \neq j$, then $(e_n)_{n\in A}$ is an *orthonormal sequence.*

Theorem 2.13. *Let H be a Hilbert space and $(e_n)_{n\in\mathbb{N}} \subset H$ an orthonormal sequence. If $(a_n)_{n\in\mathbb{N}}$ is a sequence of scalars then*

$$\sum_{n\in\mathbb{N}} a_n e_n \quad \text{converges if and only if} \quad \sum_{n\in\mathbb{N}} |a_n|^2 \tag{2.4}$$

converges. When the series converges, then

$$\Big\|\sum_{n\in\mathbb{N}} a_n e_n\Big\|^2 = \sum_{n\in\mathbb{N}} |a_n|^2.$$

Proof. We start by denoting the sequence of partial sums as

$$s_N = \sum_{n=1}^{N} a_n e_n$$

for all $N \in \mathbb{N}$. If $N, M \in \mathbb{N}$ such that $N > M$ we have

$$\|s_N - s_M\|^2 = \Big\|\sum_{n=M+1}^{N} a_n e_n\Big\|^2 = \sum_{n=M+1}^{N} \|a_n e_n\|^2 = \sum_{n=M+1}^{N} |a_n|^2$$

where the second to last equality is due to Pythagoras. The previous equality implies that $(s_N)_{N\in\mathbb{N}}$ is Cauchy if and only if

$$\sum_{n=1}^{N} |a_n|^2$$

converges. Since H is complete we have (2.4).

We know, that there exists $s \in H$ such that

$$s_N = \sum_{n=1}^{N} a_n e_n \to \sum_{n=1}^{\infty} a_n e_n = s$$

when $N \to \infty$. For all $j \in \mathbb{N}$ we have

$$\langle s, e_j\rangle = \left\langle \sum_{n=1}^{\infty} a_n e_n, e_j \right\rangle$$

and by continuity of the inner product and the orthonormality of the sequence $(e_n)_{n\in\mathbb{N}}$

$$\left\langle \sum_{n=1}^{\infty} a_n e_n, e_j \right\rangle = \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} a_n e_n, e_j \right\rangle = \langle a_j e_j, e_j\rangle = a_j.$$

We use the previous equality and conclude

$$\begin{aligned}\Big\|\sum_{n\in\mathbb{N}} a_n e_n\Big\|^2 = \|s\|^2 = \Big\langle \sum_{n=1}^{\infty} a_n e_n, s \Big\rangle = \lim_{N\to\infty} \Big\langle \sum_{n=1}^{N} a_n e_n, s \Big\rangle = \lim_{N\to\infty} \sum_{n=1}^{N} \langle a_n e_n, s\rangle \\ = \sum_{n=1}^{\infty} a_n \overline{\langle s, e_n\rangle} = \sum_{n=1}^{\infty} a_n \overline{a_n} = \sum_{n=1}^{\infty} |a_n|^2,\end{aligned}$$

which finishes the proof. □

2.2 Fourier transform

Since we study the uncertainty principles of the Fourier transform, it is important to understand Fourier analysis. For this section we again mostly follow the exposition of the Fourier theory in [44]. Also the proofs omitted in this section can be found therein. We assume that the reader is familiar with the definition and basic properties of L^p-spaces. A good source to refresh ones memory on this subject is [43].

Definition 2.14. Let $f \in L^1(\mathbb{R}^d)$ where $d \geq 1$. The *Fourier transform* of f is

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} \, \mathrm{d}x$$

for any $\xi \in \mathbb{R}^d$. In some instances we also denote the Fourier transform as $\mathcal{F}(f)(\xi)$.

We use this specific definition to avoid troublesome scaling constants in most situations. The function space the transform is operated on has a great effect on the resulting space. We showcase the function spaces that are used in this thesis. These spaces luckily happen to be the most classical examples to build the Fourier theory on. This is because of the useful properties and the wide range of applications of the spaces and also the interesting effects the Fourier transform has on them.

The natural starting point is the space of integrable functions. By simply noticing

$$\|\hat{f}\|_{L^\infty} \leq \int_{\mathbb{R}^d} |f(x)| \, \mathrm{d}x = \|f\|_{L^1}$$

we have the fact that the Fourier transform can be defined for all $\xi \in \mathbb{R}^d$ and maps the space L^1 to the space L^∞. We can actually refine our knowledge of the image of the transform with the following theorem:

Theorem 2.15. *Let $f \in L^1(\mathbb{R}^d)$. Then $\hat{f}$ is continuous and bounded function for which $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

It is good to note that the space of continuous bounded functions that vanish at infinity equipped with the sup-norm is a Banach space.

One of the great properties of the Fourier transform is that it simplifies working with some troublesome operators. This makes it an extremely useful tool to use for example with partial differential equations. The next theorems showcase the interplay between differentation and the Fourier transform.

Theorem 2.16. *Let $f \in L^1(\mathbb{R}^d)$ and $j = 1, 2, \ldots, d$. If $|f(x)| \lesssim \frac{1}{1+|x|^d}$ and $\partial_j f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then we have*

$$\left(\widehat{\frac{\partial f}{\partial x_j}}\right)(\xi) = (-i2\pi x_j f)\hat{}(\xi). \tag{2.5}$$

Theorem 2.17. *Let $f \in L^1(\mathbb{R}^d)$. If for $g(x) = xf(x)$ we have $g \in L^1(\mathbb{R}^d)$, then $f \in C^1(\mathbb{R}^d)$ and*

$$\frac{\partial}{\partial \xi_j}\hat{f}(\xi) = (-i2\pi x_j f)\hat{}(\xi). \tag{2.6}$$

From these theorems we see that the smoothness of a function depends on the rate of decay of its Fourier transform. There are many more instances of similar dualities and symmetries, for example on the Fourier side taking a convolution of two function turns into multiplication, more precisely $\mathcal{F}(f * g) = \hat{f}\hat{g}$. Two of these we will need later on:

Theorem 2.18. *Let $f \in L^1(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$:*

- *If $f_a(x) = f(x+a)$, then for all $\xi \in \mathbb{R}^d$ we have $\hat{f}_a(\xi) = e^{2\pi i a\cdot\xi}\hat{f}(\xi)$.*
- *If $f_a(x) = e^{2\pi i a\cdot x} f(x)$, then for all $\xi \in \mathbb{R}^d$ we have $\hat{f}_a(\xi) = \hat{f}(\xi - a)$.*

Before expanding the Fourier theory to other function spaces we state how a function $f \in L^1(\mathbb{R}^d)$ such that also $\hat{f} \in L^1(\mathbb{R}^d)$ can be uniquely determined by the transform.

Theorem 2.19 (Fourier inversion theorem)**.** *Let $f \in L^1(\mathbb{R}^d)$ be such that also $\hat{f} \in L^1(\mathbb{R}^d)$. Then*

$$f(x) = \int\limits_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi$$

for $x \in \mathbb{R}^d$.

Using the Fourier inversion theorem we easily get the injectivity of $\mathcal{F}$. Since the transform is a linear operator we just need to check for which $f \in L^1(\mathbb{R}^d)$ we have $\hat{f} = 0$. In this case both the function and the transform are L^1-functions so by Fourier inversion we get that $f = 0$.

From the inversion theorem also rises a problem of sorts for the Fourier L^1-theory. If we want to utilize the whole of the theory, in other words require $f, \hat{f} \in L^1$, we get that both f and $\hat{f}$ are continuous. This is due to the fact that by Fourier inversion

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, \mathrm{d}\xi = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{-2\pi i (-x) \cdot \xi} \, \mathrm{d}\xi = \mathcal{F}(\mathcal{F}(f))(-x)$$

making f a Fourier transform of a L^1-function and therefore continuous. If we want to form some sort of generalized theory that accepts functions that are not continuous, we need to look elsewhere than the space $L^1(\mathbb{R}^d)$. For this end we actually start with a space of even smoother functions.

Definition 2.20. *The Schwartz space of rapidly decreasing functions* is defined as

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d) : \sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^N |\partial^\alpha f(x)| < \infty, N \in \mathbb{N} \right\}.$$

There is no natural norm for this space, but it is possible to construct a metric with a countable family $\{p_N\}$ of norms

$$p_N(f) = \sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^N |\partial^\alpha f(x)|.$$

More details on the metric and the proof of the following results are again found in [44].

Theorem 2.21. *The metric space $(\mathcal{S}(\mathbb{R}^d), \rho)$ is complete.*

The space $\mathcal{S}(\mathbb{R}^d)$ is closed under differentation, multiplication with any polynomial, and most importantly for our purposes, the Fourier transform. The Fourier transform actually characterizes the Schwartz space.

Theorem 2.22. *The Fourier transform $\mathcal{F} \colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a continuous linear bijection.*

Since clearly $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, the Fourier inversion theorem and statements (2.5), (2.6) hold. Actually by the fact that $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ we know that for all $1 \le p < \infty$ the space $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ which makes it handy to build the L^2-Fourier theory with. For this we still need a helping hand from the following theorem.

Theorem 2.23. *Let X be a normed space, $M \subset X$ is a subspace, Y is Banach space and $T: M \to Y$ is a continuous linear mapping. There exists a unique continuous extension $T_{ex}: \overline{M} \to Y$ such that*

1. $T_{ex}x = Tx$ *for all* $x \in M$
2. $\|T_{ex}\| = \|T\|$.

Proof. We show that for all $x \in \overline{M}$, we can uniquely define the extension by taking an arbitrary sequence $(x_n) \subset M$ for which $x_n \to x$ and setting $T_{ex}x = \lim_{n\to\infty} Tx_n$.

First of all, for all $x \in \overline{M}$ there exists a sequence (x_n) such that $x_n \to x$ since x is in the closure of M. The existence of the limit of (Tx_n) can be argued by the fact that for all $m, n \in \mathbb{N}$

$$\|Tx_n - Tx_m\|_Y = \|T(x_n - x_m)\|_Y \leq \|T\|\|x_n - x_m\|_Y$$

making (Tx_n) a Cauchy sequence due to the convergence of (x_n). By assumption Y is complete, so there exists $y \in Y$ such that $Tx_n \to y$.

We need to check that this limit is independent of the choice of approximating sequence of x. For this end let $(z_n) \subset M$ be another sequence for which $z_n \to x$. Similarly as before we find a limit y_0 for (Tz_n). The triangle inequality gives us $\|x_n - z_n\|_X \leq \|x_n - x\|_X + \|x - z_n\|_X \to 0$, which implies that the sequence $(x_n - z_n)$ converges to 0 and by the continuity of T we have

$$0 = T0 = \lim_{n\to\infty} T(x_n - z_n) = \lim_{n\to\infty} Tx_n - \lim_{n\to\infty} Tz_n = y - y_0$$

giving us that $y = y_0$ and showing that for all $x \in \overline{M}$ the definition $T_{ex}x := \lim_{n\to\infty} T(x_n)$ where $(x_n) \subset M$ and $x_n \to x$ is well-defined.

The equality $T_{ex}x = Tx$ for all $x \in M$ follows from taking a sequence (x_n) where $x_n = x$ for all $n \in \mathbb{N}$. By definition $T_{ex} = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tx = Tx$.

To show the linearity of T_{ex} we take $x, y \in \overline{M}$, the approximating sequences $(x_n) \subset M$ and $(y_n) \subset M$ respectively, and scalars $a, b \in \mathbb{K}$. Since M is a subspace the sequence $(ax_n + by_n)$ is in M and converges to $(ax + by)$ so we can again say

$$\begin{aligned} T_{ex}(ax+by) &= \lim_{n\to\infty} T(ax_n + by_n) = \lim_{n\to\infty} aTx_n + \lim_{n\to\infty} bTy_n = a\lim_{n\to\infty} Tx_n + b\lim_{n\to\infty} Ty_n \\ &= aT_{ex}x + bT_{ex}y \end{aligned}$$

confirming that T_{ex} is linear.

The next thing to show is that the extension preserves the operator norm. For an arbitrary $x \in \overline{M}$ we again choose an approximating sequence $(x_n) \subset M$ and argue by the continuity of the norm mapping that

$$\|T_{ex}x\|_Y = \lim_{n\to\infty} \|Tx_n\|_Y \leq \lim_{n\to\infty} \|T\|\|x_n\|_Y = \|T\|\|x\|_Y$$

giving us $\|T_{ex}\| \leq \|T\|$. For the converse we pick an arbitrary $x \in M$ and by definition of T_{ex} get

$$\|Tx\|_Y = \|T_{ex}x\|_Y \leq \|T_{ex}\|\|x\|_Y$$

thus concluding $\|T\| \leq \|T_{ex}\|$ and therefore $\|T_{ex}\| = \|T\|$.

Finally we show that this extension is unique. Let S_{ex} be another continuous linear extension of T to $\overline{M}$, in other words $S_{ex}x = Tx$ for all $x \in M$. By assumption S_{ex} is continuous, so if we consider the continuous mapping $S_{ex} - T_{ex}$ and an approximating sequence $(x_n) \subset M$ of $x \in \overline{M}$ we have

$$\begin{aligned} S_{ex}x - T_{ex}x &= (S_{ex} - T_{ex})x \\ &= \lim_{n\to\infty} (S_{ex} - T_{ex})x_n \\ &= \lim_{n\to\infty} (S_{ex}x_n - T_{ex}x_n) \\ &= \lim_{n\to\infty} (Tx_n - Tx_n) = 0 \end{aligned}$$

concluding that $S_{ex} = T_{ex}$ and finalizing the proof. □

Theorem 2.24. *There exists a bijective linear isometry $\mathcal{F}_0 \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ that is uniquely determined by*

$$\mathcal{F}_0 f = \mathcal{F} f$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. The aim is to use the denseness of the Schwartz space to extend the Fourier transform continuously to L^2.

For $f, g \in \mathcal{S}(\mathbb{R}^d)$ we use Fourier inversion and Fubini to get

$$\begin{aligned} \langle f, g\rangle_{L^2} &= \int\limits_{\mathbb{R}^d} f(x)\overline{g(x)}\,\mathrm{d}x = \int\limits_{\mathbb{R}^d} \overline{g(x)} \int\limits_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi\,\mathrm{d}x = \int\limits_{\mathbb{R}^d} \hat{f}(\xi) \int\limits_{\mathbb{R}^d} \overline{g(x)}e^{2\pi i x\cdot\xi}\,\mathrm{d}x\,\mathrm{d}\xi \\ &= \int\limits_{\mathbb{R}^d} \hat{f}(\xi)\overline{\int\limits_{\mathbb{R}^d} g(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x}\,\mathrm{d}\xi = \int\limits_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi = \langle \hat{f}, \hat{g}\rangle_{L^2} \end{aligned}$$

thus obtaining the Parseval formula for functions in the Schwartz space. By plugging in $g = f$ we also get L^2-isometry of the Fourier transform of Schwartz functions.

Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, for every $f \in L^2(\mathbb{R}^d)$ there exists $(f_n) \subset \mathcal{S}(\mathbb{R}^d)$ such that $f_n \to f$ in L^2. We can use this to continuously extend $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ to $\mathcal{F}_0\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by setting $\mathcal{F}_0 f = \lim_{n\to\infty} \mathcal{F} f$. By Theorem 2.23 this is well-defined, uniquely determined and coincides with $\mathcal{F}$ in $\mathcal{S}(\mathbb{R}^d)$. We can extend the Plancherel identity to the whole of L^2 by noticing that

$$\|\mathcal{F}_0 f\|_{L^2} = \|\lim_{n\to\infty} \mathcal{F} f_n\|_{L^2} = \lim_{n\to\infty} \|\mathcal{F} f_n\|_{L^2} = \lim_{n\to\infty} \|f_n\|_{L^2} = \|f\|_{L^2} \tag{2.7}$$

where the second and fourth equality come from dominated convergence and the third from Plancherel of the Schwartz functions.

We also acquire Fourier inversion for L^2 by extending the Fourier inversion of Schwartz functions. The continuity of $\mathcal{F}^{-1}\colon \mathcal{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a consequence of Plancherel:

$$\|\mathcal{F}^{-1} f\|_{L^2} = \|\mathcal{F}\mathcal{F}^{-1} f\|_{L^2} = \|f\|_{L^2}.$$

Again by Theorem 2.23 the extended continuous linear mapping $\mathcal{F}_0^{-1}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined with the approximating sequences $(f_n) \subset \mathcal{S}(\mathbb{R}^d)$ is well-defined, unique, coincides with $\mathcal{F}^{-1}$ in $\mathcal{S}(\mathbb{R}^d)$ and by similar deduction as (2.7) also preserves the norm.

Using the inverse we can conclude the bijectivity with the equalities

$$\mathcal{F}_0^{-1}(\mathcal{F}_0 f) = \mathcal{F}_0^{-1}\left(\lim_{n\to\infty} \mathcal{F} f_n\right) = \lim_{n\to\infty} \mathcal{F}^{-1}\mathcal{F} f_n = \lim_{n\to\infty} f_n = f$$

and

$$\mathcal{F}_0(\mathcal{F}_0^{-1} f) = \mathcal{F}_0\left(\lim_{n\to\infty} \mathcal{F}^{-1} f_n\right) = \lim_{n\to\infty} \mathcal{F}\mathcal{F}^{-1} f_n = \lim_{n\to\infty} f_n = f.$$

The operator $\mathcal{F}_0$ is an extension that fulfills all the wanted conditions. □

We conclude this section by proving the Poisson summation formula.

Theorem 2.25 (Poisson summation formula)**.** *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a continuous function such that for all $x, \xi \in \mathbb{R}$*

$$|f(x)| \lesssim (1 + |x|)^{-1-\delta} \quad \text{and} \quad |\hat{f}(\xi)| \lesssim (1 + |\xi|)^{-1-\delta}$$

for some $\delta > 0$. Then

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i x n}$$

for all $x \in \mathbb{R}$.

Proof. We first note that the assumed upper bound of f implies $f \in L^1(\mathbb{R})$. This in turn tells us that the Fourier transform $\hat{f}$ is continuous. Therefore we may talk about pointwise values of $\hat{f}$ in good conscience.

We define the function $g\colon [0,1] \to \mathbb{C}$ by setting

$$g(x) = \sum_{k\in\mathbb{Z}} f(x+k).$$

By the upper bound of f, the series converges both absolutely and uniformly, hence g is continuous. It clearly is also 1-periodic. We calculate its Fourier coefficients:

$$\begin{aligned}\hat{g}(n) &= \int_0^1 g(x)e^{-2\pi inx}\,\mathrm{d}x = \sum_{k\in\mathbb{Z}} \int_0^1 f(x+k)e^{-2\pi inx}\,\mathrm{d}x = \sum_{k\in\mathbb{Z}} \int_k^{k+1} f(x)e^{-2\pi inx}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)e^{-2\pi inx}\,\mathrm{d}x = \hat{f}(n).\end{aligned}$$

The assumed upper bound of $\hat{f}$ combined with $\hat{g}(n) = \hat{f}(n)$ implies that the Fourier series of g converges absolutely. Therefore

$$g(x) = \sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi inx} = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$$

and by the definition of g

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi ixn}.$$

□

2.3 Sobolev spaces

We briefly discuss a very useful space of functions with a generalized concept of derivative. These Sobolev spaces are incredibly useful when working with partial differential equation. We first give a general definition of Sobolev spaces but then focus on a specific space that has useful Hilbert structure. This section is based on the lecture material [27], which also is a good resource for a reader that wants a more thorough understanding.

Before we can define the Sobolev spaces, we need to understand the weak derivative.

Definition 2.26. Let f be a function in $L^1_{loc}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is open and $\alpha \in \mathbb{N}^d$ be a multi-index. The function $g \in L^1_{loc}$ is called the *αth weak derivative of f* if we have

$$\int_\Omega f D^\alpha \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g\varphi \,\mathrm{d}x$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

The idea of a weak derivative is to extend the notion of derivative to functions that do not allow pointwise properties, while preserving as much of the uniform properties of the classical derivative.

Theorem 2.27. *Let $f \in L^1_{loc}(\Omega)$ be a function with αth weak derivative. Then this weak derivative is uniquely determined almost everywhere.*

Proof. Let $g_1, g_2 \in L^1_{loc}(\Omega)$ be αth weak derivatives of f. By definition

$$\int_\Omega g_1 \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi \, \mathrm{d}x = \int_\Omega g_2 \varphi \, \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)$. That means

$$\int_\Omega (g_1 - g_2) \varphi \, \mathrm{d}x = 0$$

for all test functions φ. It is well known that this implies that $g_1 = g_2$ almost everywhere. $\square$

Uniqueness of the weak derivative is in itself valuable, but it also means that for functions that have a classical derivative, the derivative coincides with the weak derivative.

Now that we have the weak derivative to work with we can define the Sobolev space.

Definition 2.28. Let $\Omega \subset \mathbb{R}^d$ be open. The *Sobolev space* $W_p^k(\Omega)$ is defined as

$$W_p^k(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), |\alpha| \leq k\}.$$

The norm of $W_p^k(\Omega)$ is

$$\|f\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha f|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

$$\|f\|_{W_\infty^k(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty}.$$

It is good to note that we can also use an equivalent norm

$$\|f\|_{W_p^k(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}.$$

Similarly to L^p-spaces, the elements of Sobolev spaces are in reality equivalence classes of functions that agree almost everywhere. This will not pose any problems with the handling of these spaces, especially for $W_2^1(\mathbb{R})$ as we will discuss later.

An important question when it comes to function spaces is the question of completeness. For Sobolev spaces the completeness follows quite nicely from the completeness of the L^p-spaces.

Theorem 2.29. *For $1 \leq p \leq \infty$ the space $W_p^k(\Omega)$ is a Banach space.*

We give a short sketch of the proof. For a more complete one, the reader can again turn to [27].

Proof. Let (f_n) be a Cauchy sequence in $W_p^k(\Omega)$. By the definition of the norm of $W_p^k(\Omega)$, we have for all multi-index $\alpha \in \mathbb{N}^d$ for which $|\alpha| \leq k$ that

$$\|D^\alpha f_n - D^\alpha f_m\|_{L^p(\Omega)} \leq \|f_n - f_m\|_{W_p^k(\Omega)},$$

making $(D^\alpha f_n)$ a Cauchy sequence in $L^p(\Omega)$ for all $|\alpha| \leq k$. By completeness of $L^p(\Omega)$, for every $|\alpha| \leq k$ there exists a $f_\alpha \in L^p(\Omega)$, such that $D^\alpha f_n \to f_\alpha$ in $L^p(\Omega)$.

We would like the limits f_α to be consistent with the weak derivative, in other words to be the weak derivatives of the limit f of (f_n) in $L^p(\Omega)$. With an argument utilizing the weak derivative we have for every test function $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega f D^\alpha \varphi \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega f_n D^\alpha \varphi \,\mathrm{d}x = \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha f_n \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f_\alpha \varphi \,\mathrm{d}x$$

giving us $f_\alpha = D^\alpha f$. The first and third equality can be justified with a simple use of Hölder inequality. Now for all $|\alpha| \leq k$ we have $D^\alpha f_n \to f_\alpha = D^\alpha f \in L^p(\Omega)$ so

$$\|f_n - f\|_{W_p^k(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f_n - D^\alpha f\|_{L^p} \to 0$$

making $W_p^k(\Omega)$ complete. $\square$

A perceptive reader might wonder, if the case $p = 2$ is somehow special, like with the L^p-spaces. This actually is true, and similarly to the space L^2, the Sobolev space W_2^k is a Hilbert space with the inner product

$$\langle f, g\rangle_{W_2^k(\mathbb{R}^d)} = \sum_{|\alpha|\leq k} \langle D^\alpha f, D^\alpha g\rangle_{L^2}.$$

It is evident that this inner product induces the norm. We focus on the spaces $W_2^1(I) := H^1(I)$, where $I = (a, b)$ is an open, possibly unbounded, interval. This is because we will be working with them in the chapter 5. The proofs that we show do not translate to higher dimensions, but the reader can again find the proofs in higher dimensions in [27].

For $H^1(I)$, the aim is to show that for every equivalence class in the space, there is continuous representative. Furthermore, every $f \in H^1(I)$ is differentiable almost everywhere. To this end, we recall the definition of absolutely continuous functions and some of their most important properties. The readers wanting more details we again refer to [43].

Definition 2.30. Let f be a function on an interval $[a,b]$. If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all finite collections of disjoint intervals $\{(a_n, b_n) \subset [a,b] : 1 \leq n \leq N\}$ for which

$$\sum_{n=1}^{N} |b_n - a_n| < \delta, \quad \text{we have} \quad \sum_{n=1}^{N} |f(b_n) - f(a_n)| < \varepsilon,$$

we call f *absolutely continuous.*

We immediately note that the definition implies that all absolutely continuous functions are uniformly continuous on $[a,b]$. The next theorems demonstrate how the fundamental theorem of calculus has an analogous version for the Lebesgue integral.

Theorem 2.31. *Let $f \in L^1(a,b)$. Then the function*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is absolutely continuous on $[a,b]$.

Theorem 2.32. *Let f be absolutely continuous on $[a,b]$. Then the classical derivative f' exists almost everywhere in $[a,b]$. Furthermore f' is also integrable and*

$$f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t.$$

Next we focus on proving the existence of an absolutely continuous representative. The end of this chapter is based on [8].

Lemma 2.1. *Let $f \in L^1_{loc}(I)$. If for every $\varphi \in C_0^\infty(I)$*

$$\int_I f\varphi'\,\mathrm{d}x = 0, \tag{2.8}$$

then $f = c$ almost everywhere for some constant c.

Proof. Let $\phi \in C_0^\infty(I)$. The idea is to construct a suitable test function to utilize (2.8). We first choose $\psi \in C_0^\infty(I)$ for which $\int_I \psi \,\mathrm{d}x = 1$ and define

$$g = \phi - \psi \int_I \phi \,\mathrm{d}x.$$

Since ϕ and ψ are test functions, we know that g is smooth and also compactly supported. Additionally

$$\int_I g \,\mathrm{d}t = \int_I \phi \,\mathrm{d}t - \int_I \psi \int_I \phi \,\mathrm{d}x \,\mathrm{d}t = \int_I \phi \,\mathrm{d}t - \int_I \phi \,\mathrm{d}x \int_I \psi \,\mathrm{d}t = 0. \tag{2.9}$$

We then define

$$\varphi(x) = \int_a^x g \,\mathrm{d}t.$$

Since g is compactly supported and smooth, we have $\varphi'(x) = g(x)$ for all $x \in I$. Furthermore the compact support of g implies that there exists $a_0 \in \mathbb{R}$ such that $\varphi(x) = 0$ for all $a < x \leq a_0$. Finally by combining the compact support of g with (2.9) there also exists $b_0 \in \mathbb{R}$ such that $\varphi(x) = 0$ for all $b > x \geq b_0$. Thus $g = \varphi'$, where $\varphi \in C_0^\infty(I)$.

By using (2.8) we have

$$\begin{aligned}
0 = \int_I f(t)\varphi'(t) \,\mathrm{d}t &= \int_I f(t) \left(\phi(t) - \psi(t) \int_I \phi(x) \,\mathrm{d}x \right) \mathrm{d}t \\
&= \int_I f(t)\phi(t) \,\mathrm{d}t - \int_I \int_I f(t)\psi(t)\phi(x) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_I f(t)\phi(t) \,\mathrm{d}t - \int_I \phi(t) \int_I f(x)\psi(x) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_I \phi(t) \left(f(t) - \int_I f(x)\psi(x) \,\mathrm{d}x \right) \mathrm{d}t,
\end{aligned}$$

and since ϕ was an arbitrary test function,

$$f = \int_I f(x)\psi(x) \,\mathrm{d}x =: c$$

almost everywhere, which finishes the proof. $\square$

The statement of the lemma can be interpreted as saying that if the weak derivative of the function f is zero almost everywhere the function f is constant almost everywhere, again showing how the weak derivative has analogous properties to the classical derivative.

Lemma 2.2. *Let $f \in L^1_{loc}(I)$. If we fix $x_0 \in I$ and define*

$$g(x) = \int_{x_0}^{x} f(t)\,\mathrm{d}t,$$

then

$$\int_I g\varphi'\,\mathrm{d}x = -\int_I f\varphi\,\mathrm{d}x$$

for every $\varphi \in C_0^\infty(I)$.

Proof. We start by opening up the integral

$$\begin{aligned}\int_I g(x)\varphi'(x)\,\mathrm{d}x &= \int_I \varphi'(x)\int_{x_0}^{x} f(t)\,\mathrm{d}t\,\mathrm{d}x = -\int_a^{x_0}\int_x^{x_0} \varphi'(x)f(t)\,\mathrm{d}t\,\mathrm{d}x + \int_{x_0}^{b}\int_{x_0}^{x} \varphi'(x)f(t)\,\mathrm{d}t\,\mathrm{d}x \\ &= -\int_a^{x_0}\int_a^{x_0} \chi_{[x,x_0]}(t)\varphi'(x)f(t)\,\mathrm{d}t\,\mathrm{d}x + \int_{x_0}^{b}\int_{x_0}^{b} \chi_{[x_0,x]}(t)\varphi'(x)f(t)\,\mathrm{d}t\,\mathrm{d}x\end{aligned}$$

and then applying Fubini to get

$$\begin{aligned}\int_I g(x)\varphi'(x)\,\mathrm{d}x &= -\int_a^{x_0} f(t)\int_a^{x_0} \chi_{[x,x_0]}(t)\varphi'(x)\,\mathrm{d}x\,\mathrm{d}t + \int_{x_0}^{b} f(t)\int_{x_0}^{b} \chi_{[x_0,x]}(t)\varphi'(x)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_a^{x_0} f(t)\int_a^{t} \varphi'(x)\,\mathrm{d}x\,\mathrm{d}t + \int_{x_0}^{b} f(t)\int_t^{b} \varphi'(x)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

where the last equality comes from the fact that $\chi_{[x,x_0]}(t) \neq 0$ only when $x \leq t$ and $\chi_{[x_0,x]}(t) \neq 0$ only when $x \geq t$. Since φ is smooth and compactly supported, we can utilize the fundamental theorem of calculus to say that

$$\int_a^t \varphi'(x)\,\mathrm{d}x = \varphi(t) \qquad \text{and} \qquad \int_t^b \varphi'(x)\,\mathrm{d}x = -\varphi(t),$$

thus giving us the conclusion

$$\int_I g(t)\varphi'(t)\,\mathrm{d}t = -\int_I f(t)\varphi(t)\,\mathrm{d}t,$$

which finishes the proof. $\square$

With this lemma we get an analogous statement of the fundamental theorem of calculus to the weak derivative. Now for the existence of an absolutely continuous representative for every W_2^1-function on every bounded interval.

Theorem 2.33. *Let $f \in W_2^1(I)$. There exists a function f_0 that is absolutely continuous on every interval $[c,d] \subset I$ such that $f = f_0$ almost everywhere. Moreover, f_0 is Hölder continuous with exponent $\alpha = \frac{1}{2}$.*

Proof. We fix $x_0 \in I$ and define

$$\tilde{f}(x) = \int_{x_0}^{x} f'(t)\,\mathrm{d}t.$$

We can immediately note that $\tilde{f}$ is absolutely continuous on any interval $[c,d] \subset I$ due to Theorem 2.31. By Lemma 2.2 and the definition of the weak derivative

$$\int_I \tilde{f}\varphi'\,\mathrm{d}x = -\int_I f'\varphi\,\mathrm{d}x = \int_I f\varphi'\,\mathrm{d}x$$

for every test function φ. This is equivalent with

$$\int_I (\tilde{f} - f)\varphi'\,\mathrm{d}x = 0$$

and by Lemma 2.1 there exists a constant $c \in \mathbb{R}$ such that $\tilde{f} - f = c$. The function $f_0 = \tilde{f} - c$ is absolutely continuous on any interval $[c,d] \subset I$ and $f_0 = f$ almost everywhere.

The Hölder continuity comes by applying Theorem 2.32 and Cauchy-Schwarz:

$$|f_0(x) - f_0(y)| \leq \int_x^y |f_0'(t)|\,\mathrm{d}t \leq |x-y|^{\frac{1}{2}}\|f_0'\|_{L^2(I)}$$

for all $x, y \in I$ which finishes the proof. □

With the main goal achieved, we end this section by giving two miscellaneous properties of Sobolev spaces. The proof of the first property requires some help from a lemma.

Lemma 2.3. *Let f, g be absolutely continuous on $[a,b]$. Then*

$$\int_a^x fg'\,\mathrm{d}t = f(x)g(x) - f(a)g(a) - \int_a^x f'g\,\mathrm{d}t$$

for every $x \in [a,b]$.

Proof. Let $\varphi = fg$. We first show that φ is absolutely continuous. Let $\varepsilon > 0$. There exists a $\delta > 0$ such that if $\{(a_i, b_i) : 1 \leq i \leq n\}$ is a finite collection of disjoint intervals such that $\sum_{i=1}^{n}|b_i - a_i| < \delta$, then

$$\sum_{i=1}^{n}|f(b_i) - f(a_i)| < \varepsilon, \quad \text{and} \quad \sum_{i=1}^{n}|g(b_i) - g(a_i)| < \varepsilon.$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{n}|\varphi(b_i) - \varphi(a_i)| &= \sum_{i=1}^{n}|f(b_i)g(b_i) - f(a_i)g(a_i)| \\
&= \sum_{i=1}^{n}|f(b_i)g(b_i) - f(a_i)g(b_i) + f(a_i)g(b_i) - f(a_i)g(a_i)| \\
&\leq \sum_{i=1}^{n}|g(b_i)||f(b_i) - f(a_i)| + \sum_{i=1}^{n}|f(a_i)||g(b_i) - g(a_i)| \\
&\leq (\|g\|_\infty + \|f\|_\infty)\varepsilon,
\end{aligned}$$

and we may conclude that φ is absolutely continuous. By Theorem 2.32 the derivative φ' exists almost everywhere and

$$f(x)g(x) - f(a)g(a) = \int_a^x (fg)' \,\mathrm{d}t = \int_a^x fg' \,\mathrm{d}t + \int_a^x f'g \,\mathrm{d}t.$$

A rearrangement of the previous equation finishes the proof. $\square$

Theorem 2.34. *Let $f_1 \in H^1(a,b)$ and $f_2 \in H^1(b,c)$. If*

$$d := \lim_{x \to b^-} f_1(x) = \lim_{x \to b^+} f_2(x)$$

and we set

$$f(x) = \begin{cases} f_1(x), & x \in (a,b) \\ f_2(x), & x \in (b,c) \\ d, & x = b \end{cases}$$

then $f \in H^1(a,c)$.

Proof. Let $\varphi \in C_0^\infty(a,c)$. By summing f with a suitable constant, we may assume $d = 0$. We set

$$g(x) = \begin{cases} f_1'(x), & x \in (a,b) \\ f_2'(x), & x \in (b,c) \end{cases}$$

and use Lemma 2.3 to obtain

$$\begin{aligned}
\int_a^c f\varphi' \,\mathrm{d}t &= \int_a^b f_1\varphi' \,\mathrm{d}t + \int_b^c f_2\varphi' \,\mathrm{d}t = \lim_{x\to b^-} \int_a^x f_1\varphi' \,\mathrm{d}t + \lim_{x\to b^+} \int_x^c f_2\varphi' \,\mathrm{d}t \\
&= \lim_{x\to b^-} \left(f_1(x)\varphi'(x) - f_1(a)\varphi'(a) - \int_a^x f_1'\varphi \,\mathrm{d}t \right) \\
&\quad + \lim_{x\to b^+} \left(f_2(c)\varphi'(c) - f_2(x)\varphi'(x) - \int_x^c f_2'\varphi \,\mathrm{d}t \right) \\
&= -\int_a^b f_1'\varphi \,\mathrm{d}t - \int_b^c f_2'\varphi \,\mathrm{d}t = -\int_a^c g\varphi \,\mathrm{d}t.
\end{aligned}$$

This proves that g is the weak derivative of f. Since also by definition $g \in L^2(a,c)$, we have $f \in H^1(a,c)$. $\square$

Let $\Omega \subset \mathbb{R}$ be an open set. We conclude this section by giving an alternative formulation of the norm of $H^1(\Omega)$

$$\begin{aligned}
\|f\|_{H^1(\Omega)} &= \left(\int_\Omega |f(x)|^2 \,\mathrm{d}x + \int_\Omega |f'(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} = \left(\int_\Omega |\hat{f}(\xi)|^2 \,\mathrm{d}\xi + \int_\Omega |\widehat{f'}(\xi)|^2 \,\mathrm{d}\xi \right)^{\frac{1}{2}} \\
&= \left(\int_\Omega |\hat{f}(\xi)|^2 \,\mathrm{d}\xi + \int_\Omega |2\pi i\xi \hat{f}(\xi)|^2 \,\mathrm{d}\xi \right)^{\frac{1}{2}} = \left(\int_\Omega \left(1 + 4\pi^2\xi^2\right) |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \right)^{\frac{1}{2}}
\end{aligned}$$

which we will make great use of.

2.4 Tempered distributions

When working with functions in general, taking a derivative or a Fourier transform in a classical sense is not always possible. By extending the notion of functions in a suitable way to a more general class where we still have analogous definitions of differentiation, convergence and Fourier transform, we could avoid such problems.

Definition 2.35. A linear continuous functional $T\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is called a *tempered distribution.* For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we write $T(\varphi) = \langle T, \varphi \rangle$. The space of tempered distributions is denoted as $\mathcal{S}'(\mathbb{R}^d)$.

The following theorem gives a condition for the continuity of a linear functional in $\mathcal{S}(\mathbb{R}^d)$ that is easy to work with. Recall that we denote $p_N(\varphi) = \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N |\partial^\alpha \varphi(x)|$.

Theorem 2.36. *A linear functional $T\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is continuous if and only if*

$$|\langle T, \varphi\rangle| \lesssim p_N(\varphi)$$

for some $N \in \mathbb{N}$.

The theorem follows from the construction used to define the topology in $\mathcal{S}(\mathbb{R}^d)$. Again, details are found in [44].

We define the most important operations of the tempered distributions and briefly motivate them.

Definition 2.37. Let T be a tempered distribution and $\alpha \in \mathbb{N}^d$ a multi-index. Then the *α-th distributional derivative* $D^\alpha T$ of T is defined as

$$\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha \varphi\rangle.$$

This definition makes sense, since by the definition of tempered distributions there exists a $N \in \mathbb{N}$ such that

$$|\langle T, D^\alpha \varphi\rangle| \lesssim p_N(D^\alpha \varphi) = p_{N+|\alpha|}(\varphi),$$

and thus the distributional derivative of a tempered distribution is always a tempered distribution.

Definition 2.38. Let T be a tempered distribution. If we set

$$\langle \widehat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $\widehat{T}$ is the *Fourier transform* of T.

Again the definition makes sense, since $\widehat{T}$ is continuous as a composite of two continuous operators, the Fourier transform in $\mathcal{S}(\mathbb{R}^d)$ and the functional T.

It is straightforward to show that if $f \in L^1(\mathbb{R}^d)$ is infinitely continuously differentiable, then

$$T_f(\varphi) = \int_{\mathbb{R}^d} f(x)\varphi(x)\,\mathrm{d}x, \quad \varphi \in \mathcal{S}(\mathbb{R}^d) \tag{2.10}$$

is a tempered distribution. We identify $f = T_f$ and motivate the preceding definitions. With the classical derivative, inductive use of integration by parts gives

$$\int_{\mathbb{R}^d} D^\alpha f(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_{\mathbb{R}^d} f(x)D^\alpha\varphi(x)\,\mathrm{d}x.$$

Additionally, by Fubini

$$\begin{aligned}\int_{\mathbb{R}^d} \hat{f}(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\varphi(x)e^{-2\pi ixy}\,\mathrm{d}y\,\mathrm{d}x &= \int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d} \varphi(x)e^{-2\pi ixy}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} f(y)\hat{\varphi}(y)\,\mathrm{d}y,\end{aligned}$$

hence the definitions for the distributional operations are based on their classical counterparts. We end this very brief section on tempered distributions with a definition of convergence in $\mathcal{S}'(\mathbb{R}^d)$.

Definition 2.39. Let $(T_j)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}'(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$. If $\langle T_j, \varphi\rangle \to \langle T, \varphi\rangle$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we say that T_j converges to T in $\mathcal{S}'(\mathbb{R}^d)$ and denote this by $T_j \to T$.

2.5 Heisenberg's uncertainty principle

As an appetizer we start with the most famous of the uncertainty principles, the one by Heisenberg. The actual result was not present in Heisenberg's paper [18], and the first proof was given by Kennard [26]. As this paper focuses on uncertainty principles in a purely mathematical framework we do not delve into its interpretation and significance in physics.

For a function $f \in L^2(\mathbb{R})$, Heisenberg's uncertainty principle gives a lower bound for the product of the second moments of the function and its Fourier transform. Since the second moment can be used as a quantity of how widely distributed a function is the intuitive idea of the uncertainty principle can be stated: a function and its Fourier transform cannot be simultaneously highly localized.

The proof we present here is based on the one in [13]. We need the following definition for the statement of the theorem.

Definition 2.40. A function f of the form

$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}},$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0 \neq c$, is called the *Gaussian function*.

Theorem 2.41 (Heisenberg's uncertainty principle)**.** *Let $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$. Then*

$$\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \geq \frac{\|f\|_{L^2}^4}{16\pi^2},$$

where equality holds if and only if f is the Gaussian function.

Proof. If $f = 0$ almost everywhere, the result is trivial. We can also assume that both integrals on the left-hand side are finite since otherwise the result would again be trivial. For $k, l \in \mathbb{R}$ we denote $f_{k,l}(x) = e^{2\pi i l x} f(x+k)$. By using Theorem 2.18 and a simple change of variable we get that

$$\begin{aligned}
\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi &= \int_{\mathbb{R}} x^2 |f(x+a)|^2 \, \mathrm{d}x \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi + b)|^2 \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}} x^2 |f_{a,0}(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} \xi^2 |\mathcal{F}(f_{0,-b})(\xi)|^2 \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}} x^2 |e^{2\pi i(-b)x} f_{a,0}(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} \xi^2 |e^{2\pi i a x} \mathcal{F}(f_{0,-b})(\xi)|^2 \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}} x^2 |f_{a,-b}(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} \xi^2 |\mathcal{F}(f_{a,-b})(\xi)|^2 \, \mathrm{d}\xi.
\end{aligned}$$

This means that we can assume $a = b = 0$, since if the claim holds for this case, then the previous inequality gives us

$$\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \geq \frac{\|f_{a,-b}\|_{L^2}^4}{16\pi^2} = \frac{\|f\|_{L^2}^4}{16\pi^2}$$

where $\|f_{a,-b}\|_{L^2} = \|f\|_{L^2}$ is immediate.

The assumptions $f \in L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi < \infty.$$

imply $f \in H^1(\mathbb{R})$ and moreover $f' \in L^2(\mathbb{R})$. We then obtain

$$\frac{d}{dx}|f|^2 = \frac{d}{dx}(f\overline{f}) = f'\overline{f} + f\frac{d}{dx}\overline{f} = f'\overline{f} + f\overline{f'} = 2\,\mathrm{Re}\, f\overline{f'} \tag{2.11}$$

where the last equality comes from identity $z + \bar{z} = 2 \operatorname{Re} z$. We use (2.11) in conjunction with integration by parts to get

$$
\begin{aligned}
2 \operatorname{Re} \int_s^t x f(x) \overline{f'(x)} \, \mathrm{d}x = \int_s^t 2 \operatorname{Re} x f(x) \overline{f'(x)} \, \mathrm{d}x = x|f(x)|^2 \Big|_s^t - \int_s^t |f(x)|^2 \, \mathrm{d}x \\
= t|f(t)|^2 - s|f(s)|^2 - \int_s^t |f(x)|^2 \, \mathrm{d}x
\end{aligned}
\tag{2.12}
$$

for all $-\infty < s < t < \infty$. We also use the fact that $f' \in L^2(\mathbb{R})$ and

$$
\int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x < \infty, \tag{2.13}
$$

together with Cauchy-Schwarz to conclude that the integral on the left has a finite limit when $s \to -\infty$ and $t \to \infty$.

By assumption (2.13) there exists sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that $s_n \to -\infty$ and $t_n \to \infty$ when $n \to \infty$ and also

$$
\lim_{n\to\infty} t_n^2 |f(t_n)|^2 = 0 = \lim_{n\to\infty} s_n^2 |f(s_n)|^2.
$$

Since for big enough n we have

$$
|t_n||f(t_n)|^2 < t_n^2 |f(t_n)|^2 \quad \text{and} \quad |s_n||f(s_n)|^2 < s_n^2 |f(s_n)|^2,
$$

we may conclude

$$
\lim_{n\to\infty} t_n |f(t_n)|^2 = 0 = \lim_{n\to\infty} s_n |f(s_n)|^2.
$$

We replace t and s in (2.12) with the sequences t_n and s_n respectively, and take the limit $n \to \infty$ on both sides. This yields the identity

$$
2 \operatorname{Re} \int_{\mathbb{R}} x f(x) \overline{f'(x)} \, \mathrm{d}x = - \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x
$$

and by using Cauchy-Schwarz along with Plancherel and Theorem 2.16 we obtain the inequality

$$
\begin{aligned}
\|f\|_{L^2}^4 &= 4 \left(\operatorname{Re} \int_{\mathbb{R}} x f(x) \overline{f'(x)} \, \mathrm{d}x \right)^2 \leq 4 \left(\int_{\mathbb{R}} |x f(x) \overline{f'(x)}| \, \mathrm{d}x \right)^2 \\
&\leq 4 \int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} |f'(x)|^2 \, \mathrm{d}x = 4 \int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} |\widehat{f'}(x)|^2 \, \mathrm{d}x \\
&= 16\pi^2 \int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, \mathrm{d}x.
\end{aligned}
$$

We have equality only in the case where equality holds in Cauchy-Schwarz, i.e when $f'(x) = Cxf(x)$ for some constant $C \in \mathbb{R}$. By solving the equation we get that $f(x) = Be^{\frac{Cx^2}{2}}$ where $B, C \in \mathbb{R}$. We require $f \in L^2$, so furthermore $C < 0$ must hold. $\square$

Remark. Later on we will find that the Gaussian density happens to be the extremal case for many uncertainty principles, be it explicitly or implicitly. This is partly due to the pair of facts that Gaussian functions are rapidly decaying and thus highly localized, and that the Fourier transform of a Gaussian is also a Gaussian.

Chapter 3

Uncertainty principles based on the rate of decay

We start with the historically most significant type of the uncertainty principle: how the rate of decay of a function restricts the rate of decay of its Fourier transform. In this chapter we talk about the history of uncertainty principles based on the rate of decay, and after this we present a beautiful result in this direction due to Beurling with a proof by Hedenmalm. We conclude the chapter by briefly discussing how optimal the results presented in this chapter are. But before all this we start with a fundamental example.

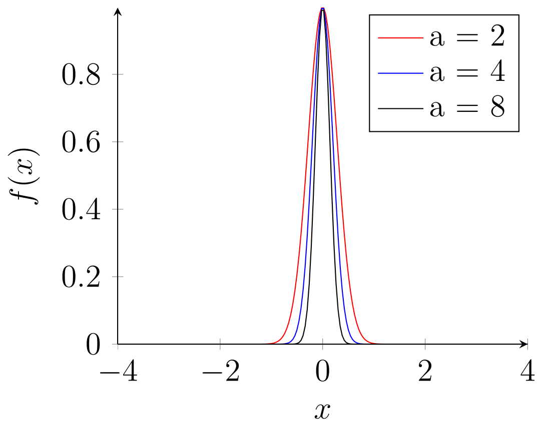
Maybe the simplest example to illustrate the phenomenon is the case of the Gaussian function $f(x) = e^{-a\pi x^2}$ where $a > 0$. The Fourier transform of f is

$$\hat{f}(\xi) = \sqrt{\frac{1}{a}} e^{-\frac{\pi}{a}\xi^2},$$

from which we can see that the bigger the constant a and therefore the more concentrated the function f, the more spread out the Fourier transform $\hat{f}$ is. This can be seen very clearly in Figure 3.1. It is also good to notice that with a choice $a = 1$ we get that $f(x) = e^{-\pi x^2}$ is an eigenfunction of the Fourier transform with an eigenvalue 1.

3.1 Classical uncertainty results on rate of decay

The history of rate of decay based uncertainty principles started with the general principle by Wiener, who remarked that a function and its Fourier transform cannot be too small at infinity. This principle was first formalized by Hardy [15] in 1933 who gave concrete limit to the simultaneous rate of decay of a function and its transform.



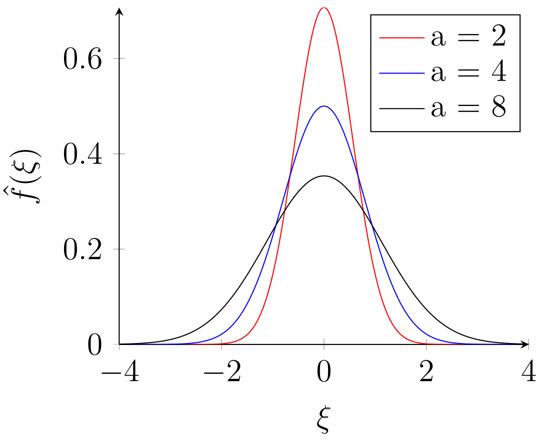


Figure 3.1: Examples of a function $f(x) = e^{-a\pi x^2}$ and its Fourier transform $\hat{f}(\xi) = \sqrt{\frac{1}{a}}e^{-\frac{\pi}{a}\xi^2}$ for different constants $a > 0$.

Theorem 3.1 (Hardy's uncertainty principle)**.** *Let f be a measurable function. Suppose*

$$|f(x)| < C|x|^n e^{-a\pi x^2} \tag{3.1}$$

$$|\hat{f}(\xi)| < C|\xi|^n e^{-b\pi\xi^2} \tag{3.2}$$

hold for some $a, b, C \in \mathbb{R}^+$ and $n \in \mathbb{N}$. If additionally $ab = 1$, then $f(x) = p(x)e^{-a\pi x^2}$ for some polynomial p for which the degree is at most n. However, if (3.1) *and* (3.2) *hold with $ab > 1$, then $f = 0$.*

It is good to note that the theorem can be often found in literature in a form where only the case $n = 0$ is considered.

This beautiful result by Hardy was quickly followed by many generalizations with similar formulation. Already the following year Ingham [21] and Morgan [35] published their own take of the principle coined by Wiener. In his article Ingham studied the possible rate of decay of a Fourier transform for a function that vanishes outside a bounded interval giving a good example of an uncertainty principle where some type of "smallness" of a function gives a restriction to a totally different type of "smallness" on the Fourier side. The result by Morgan was a more direct generalization.

Theorem 3.2 (Morgan)**.** *Let f be a measurable function and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} < 1$. Suppose*

$$|f(x)| < C|x|^n e^{-a\pi|x|^p} \tag{3.3}$$

$$|\hat{f}(\xi)| < C|\xi|^n e^{-b\pi|\xi|^q} \tag{3.4}$$

hold for some $a, b, C \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Then $f = 0$.

In the same paper Morgan gave conditions to the constants a, b such that the result can be extended to case where p and q are Hölder conjugates. Even more general formulations in the same direction were given by for example Hirschman [19], Jenkins [23], Dzhrbashyan [12] and Pfannschmidt [39].

Beurling moved towards a slightly different direction and stated a result with an integrability condition in [4]. No proof was published until 1991, when Hörmander provided one in [20] based on the notes he had made when Beurling had shown him the proof in the 1960's. To borrow Hörmander's words this result really has "a simplicity and generality which makes it very attractive".

Theorem 3.3 (Beurling)**.** *Let $f \in L^1(\mathbb{R})$. If also*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)\hat{f}(y)| e^{2\pi|xy|} \, \mathrm{d}x \, \mathrm{d}y < \infty, \tag{3.5}$$

then $f = 0$ almost everywhere.

Like the Hardy type theorems, the proof utilizes the almost magical toolbox of complex analysis, especially the Phragmén-Lindelöf theorem. It also implies the theorems by Hardy and Morgan, except of course the case $ab = 1$ in the result by Hardy. This shortcoming was fixed by the generalization of Bonami-Demange-Jaming in [6] where the assumptions were relaxed to enable the non zero solutions by Hermite functions. Also the result was generalized to higher dimensions.

To conclude this section, we prove the fact that the result by Beurling implies the ones by Hardy and Morgan. We start with the latter. Let f be a function such that (3.3) and (3.4) hold with constants a, b, C, n, p, q as described in the theorem. We aim to use suitable majorants to show that the integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)\hat{f}(y)| e^{2\pi|xy|} \, \mathrm{d}x \, \mathrm{d}y \tag{3.6}$$

is finite. First of all by (3.3) and (3.4)

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)\hat{f}(y)| e^{2\pi|xy|} \, \mathrm{d}x \, \mathrm{d}y &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |x|^n |y|^n e^{-a\pi|x|^p - b\pi|y|^q + 2\pi|xy|} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{a}{2}\pi|x|^p - \frac{b}{2}\pi|y|^q} e^{-\frac{a}{2}\pi|x|^p - \frac{b}{2}\pi|y|^q + 2\pi|xy| + \log|x|^n + \log|y|^n} \, \mathrm{d}x \, \mathrm{d}y,
\end{aligned}$$

so the integral converges if $e^{-\frac{a}{2}\pi|x|^p-\frac{b}{2}\pi|y|^q+2\pi|xy|+\log|x|^n+\log|y|^n}$ is bounded. Since $\frac{1}{p}+\frac{1}{q}<1$, there exists positive constants $\varepsilon_1, \varepsilon_2$ such that if we denote $\tilde{p}=p-\varepsilon_1$ and $\tilde{q}=q-\varepsilon_2$, then $\tilde{p}$ and $\tilde{q}$ are Hölder conjugates. By application of Young's inequality

$$|xy| \leq \frac{|x|^{\tilde{p}}}{\tilde{p}}+\frac{|y|^{\tilde{q}}}{\tilde{q}}$$

holds, from which we obtain

$$e^{-\frac{a}{2}\pi|x|^p-\frac{b}{2}\pi|y|^q+2\pi|xy|+\log|x|^n+\log|y|^n} \leq e^{-\frac{a}{2}\pi|x|^p-\frac{b}{2}\pi|y|^q+A_{\tilde{p}}|x|^{\tilde{p}}+B_{\tilde{q}}|y|^{\tilde{q}}+n\log|x|+n\log|y|},$$

where $A_{\tilde{p}}, B_{\tilde{q}}$ are constants. Since $p>\tilde{p}$ and $q>\tilde{q}$, we know that

$$e^{-\frac{a}{2}\pi|x|^p-\frac{b}{2}\pi|y|^q+A_{\tilde{p}}|x|^{\tilde{p}}+B_{\tilde{q}}|y|^{\tilde{q}}+n\log|x|+n\log|y|}$$

is bounded for all $x, y \in \mathbb{R}$ and therefore the integral (3.6) converges. By Beurling this implies $f=0$.

Proving that Beurling implies Hardy comes very similarly. Let f be a function such that (3.1) and (3.2) hold with constants $a, b, C \in \mathbb{R}^+$ and $n \in \mathbb{N}$, where $ab>1$. There exists small enough positive constants $\varepsilon_1, \varepsilon_2$ such that if we denote $\tilde{a}=a-\varepsilon_1$ and $\tilde{b}=b-\varepsilon_2$, we still have $\tilde{a}\tilde{b}>1$. We apply (3.1) and (3.2) to the integral

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}|f(x)\hat{f}(y)|e^{2\pi|xy|}\,\mathrm{d}x\,\mathrm{d}y &\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}}|x|^n|y|^n e^{-a\pi x^2-b\pi y^2+2\pi|xy|}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\varepsilon_1\pi x^2-\varepsilon_2\pi y^2}e^{-\tilde{a}\pi x^2-\tilde{b}\pi y^2+2\pi|xy|+\log|x|^n+\log|y|^n}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

and again conclude that we only need to show that $e^{-\tilde{a}\pi x^2-\tilde{b}\pi y^2+2\pi|xy|+\log|x|^n+\log|y|^n}$ is bounded to know that the integral converges. Without loss of generality we may assume $\tilde{a}>\tilde{b}$. We again choose small enough $\varepsilon>0$ such that we still have $(\tilde{a}-\varepsilon)\tilde{b}>1$, and denote $\sigma=\tilde{a}-\varepsilon$. The inequality $\tilde{b}>\frac{1}{\sigma}$ holds since otherwise we would have $(\tilde{a}-\varepsilon)\tilde{b} \leq \frac{\sigma}{\sigma}=1$, which is a contradiction. Also clearly $\tilde{a}>\sigma$. We apply Young's inequality with $x'=|x|\sqrt{\sigma}$, $y'=\frac{|y|}{\sqrt{\sigma}}$ and $p=q=2$ to obtain

$$|xy|=x'y' \leq \frac{(x')^2}{2}+\frac{(y')^2}{2}=\frac{\sigma x^2}{2}+\frac{y^2}{2\sigma}$$

which finally yields us

$$\begin{aligned}e^{-\tilde{a}\pi x^2-\tilde{b}\pi y^2+2\pi|xy|+\log|x|^n+\log|y|^n} &\leq e^{-\tilde{a}\pi x^2-\tilde{b}\pi y^2+\sigma\pi x^2+\frac{\pi}{\sigma}y^2+n\log|x|+n\log|y|} \\ &= e^{-(\tilde{a}-\sigma)\pi x^2-(\tilde{b}-\frac{1}{\sigma})\pi y^2+n\log|x|+n\log|y|}.\end{aligned}$$

Since $\tilde{a}>\sigma$ and $\tilde{b}>\frac{1}{\sigma}$, we know that $e^{-(\tilde{a}-\sigma)\pi x^2-(\tilde{b}-\frac{1}{\sigma})\pi y^2+n\log|x|+n\log|y|}$ is bounded for all $x, y \in \mathbb{R}$. Hence the integral (3.6) converges and therefore $f=0$ by Beurling.

3.2 Hedenmalm's proof of Beurling's result

In this section we focus on the result by Beurling. In 2012 Hedenmalm gave a new short proof which we will present here. Before doing just that, we briefly study the consequences of condition (3.5). This will help us show that the proof of Hedenmalm implies the result by Beurling. This whole section is based on the work of Hedenmalm in [17].

For the forthcoming discussion we need to quickly define an important complex analytic concept:

Definition 3.4. Let f be an analytic function in a domain $U \subset \mathbb{C}$. If g is an analytic function in a domain $V \subset \mathbb{C}$ such that $U \subset V$ and $g|_U = f$, we call g the *analytic continuation* of f to the domain V.

Remark. By the identity theorem the analytic continuation of a function is always unique.

Now we can start the preparation for Hedenmalm's theorem. Let us define an auxiliary function

$$F(\lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{f}(x)\hat{f}(y)e^{2\pi i\lambda xy}\,\mathrm{d}x\,\mathrm{d}y,$$

without explicitly stating the domain of the definition at this point. The function F is fundamental to the argument of the result of Hedenmalm and also ties the result to the result of Beurling.

We first look at F in the context of Beurling. Let $f \in L^1(\mathbb{R})$ be such that (3.5) holds. The condition immediately implies that F can be defined for every $\lambda \in \mathbb{R}$. By (3.5)

$$\|f\|_{L^1(\mathbb{R})}\|\hat{f}\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)\hat{f}(y)|\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)\hat{f}(y)|e^{2\pi|xy|}\,\mathrm{d}x\,\mathrm{d}y < \infty,$$

giving us $f, \hat{f} \in L^1(\mathbb{R})$ and therefore also $f, \hat{f} \in L^\infty(\mathbb{R})$. We know that functions that are essentially bounded and integrable on $\mathbb{R}$ are square integrable. Hence we may conclude that $f, \hat{f} \in L^2(\mathbb{R})$.

If we consider the strip $S := \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < 1\}$ we can say that for all $\lambda \in \overline{S}$

$$|F(\lambda)| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)\hat{f}(y)|e^{2\pi|\operatorname{Im}\lambda||xy|}\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)\hat{f}(y)|e^{2\pi|xy|}\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{3.7}$$

showing that F is well-defined in $\overline{S}$. Additionally, by dominated convergence F is continuous in the same strip. Next we take a closed, piecewise C^1 path $\gamma \subset S$ and by using the analyticity of the exponential function together with the theorems by Fubini and Cauchy we get that

$$\int_{\gamma} F(\lambda)\,\mathrm{d}\lambda = \int_{\gamma}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f}(x)\hat{f}(y)e^{2\pi i\lambda xy}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\lambda = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f}(x)\hat{f}(y)\int_{\gamma} e^{2\pi i\lambda xy}\,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y = 0.$$

By Morera's theorem we can conclude that F is analytic in S.

Before stating Hedenmalm's theorem we note that by the Fourier inversion theorem

$$\int_{\mathbb{R}} \hat{f}(y)e^{2\pi i\lambda xy}\,\mathrm{d}y = f(\lambda x)$$

for all $x, \lambda \in \mathbb{R}$, so

$$F(\lambda) = \int_{\mathbb{R}} \overline{f}(x)f(\lambda x)\,\mathrm{d}x \tag{3.8}$$

is an alternative representation for F. Also, we denote the area measure in $\mathbb{C}$ by $\mathrm{d}A(\lambda)$.

Theorem 3.5 (Hedenmalm)**.** *Suppose $f \in L^2(\mathbb{R})$ and that $F(\lambda)$ is defined as in (3.8) for $\lambda \in \mathbb{R}\setminus\{0\}$. If $F(\lambda)$ can also be analytically continued to a neighbourhood U of $\overline{B}\setminus\{\pm i\}$ so that*

$$\int_{B} |F(\lambda)|^2|\lambda^2+1|\,\mathrm{d}A(\lambda) < \infty, \tag{3.9}$$

then $F(\lambda) \equiv c_0(\lambda^2+1)^{-\frac{1}{2}}$, for some $c_0 \geq 0$. Here we use the branch of square root with the branch cut on the nonnegative real axis. Also, if $\inf_{\lambda\in B}|F(\lambda)|^2|\lambda^2+1| = 0$, then $f \equiv 0$ almost everywhere.

The preceding discussion makes it is easy to see how much weaker the assumptions are in Hedenmalm's theorem. We showed that the requirement $f \in L^1(\mathbb{R})$ together with condition (3.5) gives us $f \in L^2(\mathbb{R})$. Condition (3.5) also implies analytic continuation to a bigger domain S than in Hedenmalm's case whilst still fulfilling the requirement (3.9) because of (3.7). Finally the boundedness of F on the strip $\overline{S}$ means that $\inf_{\lambda\in B}|F(\lambda)|^2|\lambda^2+1| = 0$, and thus Hedenmalm's theorem implies the result by Beurling.

Proof. To make use of the assumption (3.9), we define the function Φ by setting $\Phi(\lambda) = \sqrt{\lambda^2+1}F(\lambda)$. Since $\sqrt{\lambda^2+1}$ is analytic in $\mathbb{C}\setminus i(\mathbb{R}\setminus(-1,1))$ and by the assumption of the analytic continuation of F, the function Φ is analytic in a neighbourhood of $\overline{B} \setminus \{\pm i\}$. Also, by a change of variable we have for all $\lambda \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned}\Phi(\lambda) &= \sqrt{\lambda^2+1}F(\lambda) = \sqrt{\lambda^2+1}\int_{\mathbb{R}} \overline{f}(x)f(\lambda x)\,\mathrm{d}x = \sqrt{\lambda^2+1}\frac{1}{|\lambda|}\int_{\mathbb{R}} \overline{f}\left(\frac{x}{\lambda}\right)f(x)\,\mathrm{d}x \\ &= \sqrt{\frac{1}{\lambda^2}+1}\overline{\int_{\mathbb{R}} \overline{f}(x)f\left(\frac{x}{\lambda}\right)\mathrm{d}x} = \sqrt{\frac{1}{\lambda^2}+1}\overline{F}\left(\frac{1}{\lambda}\right) = \overline{\Phi}\left(\frac{1}{\lambda}\right) = \overline{\Phi}\left(\frac{1}{\overline{\lambda}}\right).\end{aligned} \tag{3.10}$$

By combining the symmetry property (3.10) with the analyticity of Φ in the neighbourhood U of $\overline{B} \setminus \{\pm i\}$, we get that Φ is real analytic on the whole real line.

Next, we consider the set $U_e = \{\lambda \in \mathbb{C} : \frac{1}{\overline{\lambda}} \in U\}$. Note that U_e is a neighbourhood of $\overline{B}_e \setminus \{\pm i\}$ where B_e is the exterior of the unit disk. We define a function $g \colon U_e \mapsto \mathbb{C}$, by setting $g(\lambda) = \overline{\Phi}\left(\frac{1}{\overline{\lambda}}\right)$ and notice that by definition, g is analytic in whole of U_e and also it coincides with Φ in $U_e \cap U$. If we then define a function h in $U \cup U_e$ by

$$h(\lambda) = \begin{cases} \Phi(\lambda), & \lambda \in U \\ g(\lambda), & \lambda \in U_e \end{cases}$$

h is well-defined since $\Phi = g$ on $U \cap U_e$ and analytic since both Φ and g are analytic on their respective domains. By definition $\Phi = h$ on U and thus h is an analytic continuation of Φ to $U \cup U_e = \mathbb{C} \setminus \{\pm i\}$.

By using the assumption (3.9) with the symmetry property (3.10), now extended to $\mathbb{C} \setminus \{\pm i\}$, we get

$$\begin{aligned}\int_{\overline{B}_e} |\Phi(\lambda)|^2 \frac{1}{|\lambda|^2}\,\mathrm{d}A(\lambda) &= \int_1^\infty \int_0^{2\pi} |\Phi(re^{\theta i})|^2\,\mathrm{d}\theta\,\frac{1}{r}\,\mathrm{d}r = \int_1^\infty \int_0^{2\pi} |\overline{\Phi}\left(\frac{1}{r}e^{-\theta i}\right)|^2\,\mathrm{d}\theta\frac{1}{r}\,\mathrm{d}r \\ &= \int_0^1 \int_0^{2\pi} |\overline{\Phi}(re^{-\theta i})|^2\,\mathrm{d}\theta\, r\,\mathrm{d}r = \int_B |\Phi(\lambda)|^2\,\mathrm{d}A(\lambda) < \infty,\end{aligned} \tag{3.11}$$

and by combining the square area-integrability properties (3.9) and (3.11) we get that Φ is square area-integrable in a neighbourhood of $\{\pm i\}$. Since the function Φ is analytic in the punctured disk $B(i, R) \setminus \{i\}$, where $0 < R < 1$, it can be presented by a Laurent series. We compute the area integral by using the Laurent series

$$\begin{aligned}\int_{B(i,R)\setminus\{i\}} |\Phi(\lambda)|^2\,\mathrm{d}A(\lambda) &= \int_{B(i,R)\setminus\{i\}} \Big|\sum_{n=-\infty}^{\infty} a_n(\lambda - i)^n\Big|^2\,\mathrm{d}A(\lambda) = \int_{B(0,R)\setminus\{0\}} \Big|\sum_{n=-\infty}^{\infty} a_n\lambda^n\Big|^2\,\mathrm{d}A(\lambda) \\ &= \int_0^{2\pi}\int_0^R \Big|\sum_{n=-\infty}^{\infty} a_n r^n e^{n\theta i}\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int_0^R \Big\|\sum_{n=-\infty}^{\infty} a_n r^n e_n\Big\|^2_{L^2(0,2\pi)} r\,\mathrm{d}r,\end{aligned}$$

where the last equality is due to Tonelli, and $e_n := e^{in\theta}$. Since $(e_n)_{n\in\mathbb{Z}}$ is an orthogonal sequence in $L^2(0,2\pi)$ and the Laurent series converges for all $0 < r < R$ we can use Theorem 2.13 to say

$$2\pi \int_0^R \Big\| \sum_{n=-\infty}^{\infty} a_n r^n e_n \Big\|_{L^2(0,2\pi)}^2 r \,\mathrm{d}r = 2\pi \int_0^R \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n+1} \,\mathrm{d}r = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 \int_0^R r^{2n+1} \,\mathrm{d}r.$$

The last integral clearly diverges for any negative $n \in \mathbb{Z}$ and by square area integrability of Φ in $B(i,R) \setminus \{i\}$ we get that for all negative indices $a_n = 0$ and Φ can be analytically continued over i.

A similar argument works for $-i$, which means that Φ is an entire function. Therefore Φ is bounded on every compact set in $\mathbb{C}$. Additionally, by the symmetry property (3.10) extended to the whole complex plane

$$\lim_{|\lambda|\to\infty} |\Phi(\lambda)| = \lim_{|\lambda|\to\infty} |\overline{\Phi}\left(\frac{1}{\overline{\lambda}}\right)| = \lim_{|\lambda|\to 0} |\overline{\Phi}(\lambda)| < \infty,$$

which concludes that Φ is bounded in the whole complex plane. Hence by Liouville's theorem Φ is a constant function and from the definition of Φ we get that

$$F(\lambda) = c_0(\lambda^2+1)^{-\frac{1}{2}}.$$

Furthermore

$$c_0 = \Phi(1) = \sqrt{2} \int_{\mathbb{R}} \overline{f}(x) f(x) \,\mathrm{d}x = \sqrt{2} \int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}x \geq 0, \tag{3.12}$$

which completes the proof of the first claim.

If we assume that $\inf_{\lambda\in B} |F(\lambda)|^2 |\lambda^2+1| = 0$ we get $c_0 = 0$, and that finally gives us $F \equiv 0$, which proves the second claim. $\square$

Remark. The article also contains version of the result for higher dimensions, where the proof uses the same methods as in the one-dimensional case with some tweaks to take into account the multidimensional framework.

We show that the Gaussian density works again as the extremal case. If we set $f(x) = e^{-\alpha\pi x^2}$, where $\operatorname{Re}\alpha > 0$, then by standard computation with techniques of Gaussian integral we get that for all $\lambda \in \mathbb{R}$

$$F(\lambda) = \int_{\mathbb{R}} \overline{f}(x) f(\lambda x) \,\mathrm{d}x = \int_{\mathbb{R}} e^{-\pi(\overline{\alpha}+\alpha\lambda^2)x^2} \,\mathrm{d}x = \sqrt{\frac{\pi}{\pi(\overline{\alpha}+\alpha\lambda^2)}} = \frac{1}{\sqrt{\overline{\alpha}}} \frac{1}{\sqrt{1+\frac{\alpha}{\overline{\alpha}}\lambda^2}}.$$

It is important to notice that the integral makes sense, because $\mathrm{Re}(\pi(\overline{\alpha}+\alpha\lambda^2)) = \pi(\mathrm{Re}\,\overline{\alpha}+\lambda^2\,\mathrm{Re}\,\alpha) > 0$. We study the function

$$F(\lambda) = \frac{1}{\sqrt{\overline{\alpha}}}\frac{1}{\sqrt{1+\frac{\alpha}{\overline{\alpha}}\lambda^2}}, \tag{3.13}$$

also for complex values of λ. The function has branch points at roots of $\lambda^2 = -\frac{\overline{\alpha}}{\alpha}$. These two branch points are located on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and F_0 is analytic in the neighbourhood of the unit ball from which the branch points are removed. The only case where the function has an analytic continuation to a neighbourhood of $\overline{B} \setminus \{\pm i\}$ as in the statement of the theorem is when $\alpha = 1$. Thus $f(x) = e^{-\pi x^2}$ is again the extremal case.

Chapter 4

The effect of the support of the function on the Fourier transform

In this chapter we delve into how the size of the support of a function affects different properties of its Fourier transform. Again, the term "size" is ambiguous so we investigate a few intuitive interpretations of it. We start with perhaps the most straightforward, boundedness, and how it affects the Fourier transform when the transform is analytically continued to the complex plane. After that we quantify the size of the support in terms of the Lebesgue measure and examine how a support with finite Lebesgue measure restricts the transform. Finally we briefly talk about the case of the porous support and mostly guide an interested reader to good sources of information for a more detailed survey.

4.1 Continuing the Fourier transform to the complex plane

Although we utilized some complex methods in the last chapter, we have not explicitly studied the Fourier transform in a complex context. But often the transform can be extended to an analytic function of some domain of the complex plane. In this section we do just that and use it to show that a nontrivial compactly supported function cannot have a compactly supported Fourier transform. As a bonus we mention how in this way we can characterize two important function spaces. This section is based on the chapter about holomorphic Fourier transforms in [43].

Let us start with a function $f \in L^2(\mathbb{R})$ such that $\operatorname{supp} f \subset [-a, a]$ for some $a \in \mathbb{R}$ and $f \neq 0$. Since the support of f is bounded, we also have $f \in L^1(\mathbb{R})$. We define for all

$z \in \mathbb{C}$ the following function:

$$\hat{f}(z) = \int_{-a}^{a} f(x)e^{-2\pi ixz}\,\mathrm{d}x.$$

Since for every $z \in \mathbb{C}$

$$|\hat{f}(z)| \leq \int_{-a}^{a} |f(x)e^{-2\pi ixz}|\,\mathrm{d}x \leq \sup_{x\in[-a,a]} |e^{-2\pi ixz}|\|f\|_{L^1(\mathbb{R})} \leq e^{2\pi a|z|}\|f\|_{L^1(\mathbb{R})} < \infty,$$

we have that $\hat{f}$ can be defined for all $z \in \mathbb{C}$. We can deduce the continuity of $\hat{f}$ for all $z \in \mathbb{C}$ by noticing that

$$\lim_{h\to 0} |\hat{f}(z+h) - \hat{f}(z)| \leq \lim_{h\to 0} \int_{-a}^{a} |f(x)||e^{-2\pi ix(z+h)} - e^{-2\pi ixz}|\,\mathrm{d}x \to 0$$

where the convergence is due to dominated convergence. Then if we take a closed, piecewise C^1 path γ in $\mathbb{C}$, we may utilize Fubini's theorem to obtain

$$\int_{\gamma} \hat{f}(z)\,\mathrm{d}z = \int_{\gamma}\int_{-a}^{a} f(x)e^{-2\pi ixz}\,\mathrm{d}x\,\mathrm{d}z = \int_{-a}^{a} f(x)\int_{\gamma} e^{-2\pi ixz}\,\mathrm{d}z\,\mathrm{d}x = 0$$

where the last equality is due to the analyticity of the exponential function and Cauchy's theorem. The theorem by Morera now says that $\hat{f}$ is an entire function. The function $\hat{f}$ also clearly coincides with the Fourier transform of f on the real line and thus $\hat{f}$ is the analytic continuation of the Fourier transform of f to the entire complex plane.

Since the analytic continuation $\hat{f}$ of the Fourier transform of f is entire we can conclude that the set where the Fourier transform of f vanishes may not have an accumulation point. If it had any accumulation points, by identity theorem $\hat{f}$ would vanish, meaning that the Fourier transform would vanish as well. Obviously this implies that the support of the Fourier transform cannot be bounded.

Before moving to the next case we make an important observation. For all $\xi \in \mathbb{R}$ we have

$$|\hat{f}(\xi)| \leq \int_{-a}^{a} |f(x)e^{-2\pi ix\xi}|\,\mathrm{d}x \leq e^{2\pi a|\mathrm{Im}\,\xi|} \int_{-a}^{a} |f(x)|\,\mathrm{d}x \leq \|f\|_{L^1(\mathbb{R})}e^{2\pi a|\xi|}.$$

The entire functions for which $|f(z)| \leq Ce^{a|z|}$ for a constant $C > 0$, are said to be of *exponential type* a and the space of functions f of exponential type $2\pi a$ such that

$f \in L^2(\mathbb{R})$ is called the *Paley-Wiener space* $\mathcal{PW}_a$. The previous deduction now says that the analytic continuation of the Fourier transform of a function with a support in $[-a, a]$ belongs in the Paley-Wiener space $\mathcal{PW}_a$.

Let us next consider a function $F \in L^2(\mathbb{R})$ such that $\operatorname{supp} f \subset [0, \infty)$. This time we define a function

$$f(z) = \int_0^\infty F(x) e^{2\pi i x z} \, \mathrm{d}x,$$

where $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. This is well-defined since

$$|\hat{f}(z)| \leq \int_0^\infty |f(x) e^{2\pi i x z}| \, \mathrm{d}x = \int_0^\infty |f(x)| e^{-2\pi x \operatorname{Im} z} \, \mathrm{d}x \leq \|f\|_{L^2(\mathbb{R})} \left(\int_0^\infty e^{-4\pi x \operatorname{Im} z} \, \mathrm{d}x \right)^{\frac{1}{2}} < \infty,$$

for every $z \in \mathbb{C}^+$. We can argue continuity of f in $z \in \mathbb{C}^+$ similarly as in the previous case with the added caveat that we restrict h to such values that $\operatorname{Im}(z + h) > 0$. Furthermore we can again utilize Fubini's, Cauchy's and Morera's theorems to conclude analyticity in $\mathbb{C}^+$ with similar strategy as in previous case. Again, we notice that f coincides with the inverse Fourier transform of F on the real line and thus is the analytic continuation of it to $\mathbb{C}^+$.

Next, if we use the standard presentation $z = x + yi$ for the complex number with a fixed $y > 0$, we get

$$f(x + yi) = \int_0^\infty F(t) e^{-2\pi y t} e^{2\pi i x t} \, \mathrm{d}t \tag{4.1}$$

and we may think of $f(x + yi)$ as the inverse Fourier transform of $t \mapsto F(t) e^{-2\pi y t}$. This implies with the help of Plancherel that

$$\int_{\mathbb{R}} |f(x + yi)|^2 \, \mathrm{d}x = \int_0^\infty |F(t)|^2 e^{-4\pi y t} \, \mathrm{d}t \leq \int_0^\infty |F(t)|^2 \, \mathrm{d}t < \infty,$$

in other words that the mapping $x \mapsto f(x + yi)$ is an L^2 function for all $y > 0$. The analytic functions in $\mathbb{C}^+$ for which

$$\sup_{0 < y < \infty} \int_{\mathbb{R}} |f(x + yi)|^2 \, \mathrm{d}x < \infty$$

form the *Hardy space* of functions $H^2(\mathbb{C}^+)$ on the upper plane.

What makes these cases interesting is the fact that the converse statements also hold according to the theorems by Paley and Weiner [46]. These theorems tell us that the Paley-Wiener spaces and the Hardy space on the upper half-plane can be characterized as Fourier transform of the space of compactly supported and the space of one-sidedly supported functions respectively. Moreover, the Paley-Wiener theorem in the case of compact support was generalized to distributions on $\mathbb{R}^d$ by Schwartz [45].

4.2 The result by Benedicks

The boundedness is obviously not the only way of looking at the size of the support of a function. One very natural interpretation of size is unsurprisingly the measure, in this case the Lebesgue measure. An interesting and easy to state result in this direction is the one proved by Benedicks [2].

Theorem 4.1 (Benedicks)**.** *Let $f \in L^1(\mathbb{R}^d)$, $A = \{x \in \mathbb{R}^d : f(x) \neq 0\}$ and $B = \{\xi \in \mathbb{R}^d : \hat{f}(\xi) \neq 0\}$. If the Lebesgue measure of both A and B are finite, then $f = 0$ almost everywhere.*

Proof. We first note that without loss of generality, we can assume $m(A) < 1$. To justify this, we choose a suitable constant $a \in \mathbb{R}^d$ to define a dilation $f_a(x) := f(ax)$ such that $m(A_a) < 1$, where A_a is the set where f_a does not vanish. Since

$$\hat{f}_a(\xi) = \int\limits_{\mathbb{R}^d} f(ax)e^{-2\pi i\xi\cdot x}\,\mathrm{d}x = \frac{1}{a^d}\int\limits_{\mathbb{R}^d} f(x)e^{-2\pi i\frac{\xi}{a}\cdot x}\,\mathrm{d}x = \frac{1}{a^d}\hat{f}\left(\frac{\xi}{a}\right),$$

we know that for the set $B_a := \{\xi \in \mathbb{R}^d : \hat{f}_a(\xi) \neq 0\}$ we have

$$\begin{aligned} m(B_a) &= m(\{\xi \in \mathbb{R}^d : \hat{f}_a(\xi) \neq 0\}) = m\left(\left\{\xi \in \mathbb{R}^d : \frac{1}{a^d}\hat{f}\left(\frac{\xi}{a}\right) \neq 0\right\}\right) \\ &= m\left(\left\{\xi \in \mathbb{R}^d : \frac{\xi}{a} \in B\right\}\right) = a^d m\left(\left\{\xi \in \mathbb{R}^d : \xi \in B\right\}\right) = a^d m(B). \end{aligned}$$

This means that the dilation preserves the finiteness of the Lebesgue measure of the original set B.

We define a function

$$\tilde{\chi}_B(\xi) = \sum_{k\in\mathbb{Z}^d} \chi_B(\xi - k),$$

where χ_B is again the characteristic function of B. We can immediately state that $\tilde{\chi}_B$ is positive, measurable and 1-periodic. By an application of Fubini we get

$$
\begin{aligned}
\int\limits_{[0,1]^d} \tilde{\chi}_B(\xi)\,\mathrm{d}\xi &= \int\limits_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} \chi_B(\xi-k)\,\mathrm{d}\xi = \sum_{k\in\mathbb{Z}^d} \int\limits_{[0,1]^d} \chi_B(\xi-k)\,\mathrm{d}\xi \\
&= \sum_{k\in\mathbb{Z}^d} \int\limits_{k+[0,1]^d} \chi_B(\xi)\,\mathrm{d}\xi = \int\limits_{\mathbb{R}^d} \chi_B(\xi)\,\mathrm{d}\xi = m(B) < \infty
\end{aligned}
$$

and by pairing this with the periodicity, we get that $\tilde{\chi}_B(\xi) < \infty$ almost everywhere making $\tilde{\chi}_B$ well-defined. By construction this also implies that for almost all $\xi_0 \in \mathbb{R}^d$, we have $\chi_B(\xi_0 + k) \neq 0$ for only a finite number of $k \in \mathbb{Z}^d$. Since χ_B is the characteristic function of the set of points $\xi \in \mathbb{R}^d$ for which $\hat{f}(\xi) \neq 0$, we get that for almost all $\xi_0 \in \mathbb{R}^d$ we have $\hat{f}(\xi_0 + k) \neq 0$ for only finite number of $k \in \mathbb{Z}^d$.

To show that the Fourier transform $\hat{f}$ vanishes almost everywhere, we define for a fixed $\xi_0 \in \mathbb{R}^d$ the auxiliary function

$$
\tilde{f}_{\xi_0}(x) = \sum_{m\in\mathbb{Z}^d} e^{-i\xi_0\cdot(x-m)} f(x-m), \quad x \in \mathbb{T}^d
$$

where $\mathbb{T}^d = [0,1]^d$. The strategy is to find a link between the Fourier coefficients of $\tilde{f}_{\xi_0}$ and the Fourier transform of f and then show that $\tilde{f}_{\xi_0}$ vanishes.

We start by showing that $\tilde{f}_{\xi_0}$ has a few useful properties. First of all, since $f \in L^1(\mathbb{R}^d)$, we have

$$
\begin{aligned}
\|\tilde{f}_{\xi_0}\|_{L^1(\mathbb{T}^d)} &\leq \int\limits_{\mathbb{T}^d} \sum_{m\in\mathbb{Z}^d} |e^{-i2\pi\xi_0\cdot(x-m)} f(x-m)|\,\mathrm{d}x \\
&= \int\limits_{\mathbb{T}^d} \sum_{m\in\mathbb{Z}^d} |f(x-m)|\,\mathrm{d}x \\
&= \|f\|_{L^1(\mathbb{R}^d)} < \infty,
\end{aligned}
$$

and thus $\tilde{f}_{\xi_0}$ is well-defined with $\tilde{f}_{\xi_0} \in L^1(\mathbb{T}^d)$. Also, by using Fubini and periodicity of the exponential function, we find the representation

$$
\begin{aligned}
\left(\tilde{f}_{\xi_0}\right)\hat{}\,(k) &= \int\limits_{\mathbb{T}^d} \sum_{m\in\mathbb{Z}^d} e^{-i2\pi\xi_0\cdot(x-m)} f(x-m) e^{-i2\pi k\cdot x}\,\mathrm{d}x \\
&= \sum_{m\in\mathbb{Z}^d} \int\limits_{\mathbb{T}^d} e^{-i2\pi\xi_0\cdot(x-m)} f(x-m) e^{-i2\pi k\cdot(x-m)}\,\mathrm{d}x \\
&= \sum_{m\in\mathbb{Z}^d} \int\limits_{-m+[0,1]^d} e^{-i2\pi\xi_0\cdot x} f(x) e^{-i2\pi k\cdot x}\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}^d} e^{-i2\pi(\xi_0+k)\cdot x} f(x)\,\mathrm{d}x = \hat{f}(\xi_0 + k)
\end{aligned}
$$

for the Fourier coefficients of $\tilde{f}_{\xi_0}$. Paired with the knowledge that for almost all ξ_0, we have $\hat{f}(\xi_0 + k) \neq 0$ for only finite number of $k \in \mathbb{Z}^d$, we know that $\tilde{f}_{\xi_0}$ is a trigonometric polynomial for almost all ξ_0. Also the fact that $m(A) < 1$ implies that the measure of the set where $\tilde{f}_{\xi_0}$ does not vanish is less than 1. That means that $\tilde{f}_{\xi_0}$ vanishes in a set of positive measure. As an entire function, nonzero trigonometric polynomial can not vanish on a set of positive measure and therefore $\tilde{f}_{\xi_0} = 0$ almost everywhere and consequently for almost all $\xi_0 \in \mathbb{R}^d$ we have $\hat{f}(\xi_0 + k) = 0$ for all $k \in \mathbb{Z}^d$. From this it follows that $\hat{f} = 0$ almost everywhere which finally gives us $f = 0$ almost everywhere. □

Corollary 4.2. Theorem 4.1 extends to functions $f \in L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$.

Proof. Let $f \in L^p(\mathbb{R}^d)$. By definition of the set A and the fact that $m(A) < \infty$, we can say

$$\|f\|_{L^1(\mathbb{R}^d)} = \|f\|_{L^1(A)} \lesssim \|f\|_{L^p(A)} = \|f\|_{L^p(\mathbb{R}^d)} < \infty.$$

Since $f \in L^1(\mathbb{R}^d)$, we can apply Theorem 4.1 and the proof is complete. □

Remark. Giving examples of functions f such that f does not vanish in a set of finite measure but the Fourier transform has a gap in its support, is a nontrivial task. Here a gap in the support means an interval where a function vanishes. Kargaev-Volberg found such function and also gave examples in two other similar situations in [25].

4.3 Generalizations of the result by Benedicks

To properly discuss the ensuing results we first give some common terminology:

Definition 4.3. Let (A, B) be a pair of measurable sets in $\mathbb{R}^d$. This pair is called *weakly annihilating* if for every $f \in L^2(\mathbb{R}^d)$ such that $\operatorname{supp} f \subset A, \operatorname{supp} \hat{f} \subset B$, we have $f = 0$.

Definition 4.4. Let (A, B) be a pair of measurable sets in $\mathbb{R}^d$. This pair is called *strongly annihilating* if for every $f \in L^2(\mathbb{R}^d)$ there exists a constant $C_{A,B}$ such that

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq C_{A,B} \left(\int\limits_{A^c} |f|^2 \, \mathrm{d}x + \int\limits_{B^c} |\hat{f}|^2 \, \mathrm{d}\xi \right).$$

We immediately remark that every strongly annihilating pair is also weakly annihilating. With this terminology we can say that the statement of Benedicks's result is that any pair (A, B) of measurable sets of finite Lebesgue measure is weakly annihilating. A few years after Benedicks had finished work on the preprint, Amrein and Berthier published a generalization utilizing properties of Hilbert space [1]. Their article stated that any pair

(A, B) of measurable sets of finite measure is also strongly annihilating. This new result provides a sort of stability to the result of Benedicks. Where the result by Benedicks talks about how simultaneous vanishing of f and $\hat{f}$ outside of sets with finite Lebesgue measure forces uniform vanishing, the one by Amrein-Berthier extends it to discussion about simultaneous smallness of f and $\hat{f}$ outside of sets with finite Lebesgue measure forcing uniform smallness.

Later on Nazarov [36] made further improvements to the constant in the result of Amrein-Berthier when working in one dimension, giving us an explicit formulation $C_{A,B} = Ce^{C|A||B|}$ of the constant. Finally Jaming [22] generalized the refinement by Nazarov to higher dimension with a little geometric tinkering on the constant. For pairs of sets with finite Lebesgue measure, the trail of important results ends here but there is a wide selection of results studying the strongly annihilating pairs for more general pairs of sets. An interested reader should refer to the excellent survey by Bonami and Demange [5].

4.4 Briefly on the Gap problem

We conclude this chapter by briefly mentioning the Gap problem. An ample foundation of theory about measures would be required to fully understand the proofs so we settle with a short description of the problem and some classical and modern results. The subject is of such significance that we almost have to touch on it, if only briefly. Since this section is focused on results regarding measures, we start by defining the Fourier transform of a measure.

Definition 4.5. Let μ be a finite Borel complex measure on $\mathbb{R}$. The *Fourier transform* of μ is

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} \,\mathrm{d}\mu(x).$$

For a measure μ with given properties the aim of the Gap problem is to find a maximal sized gap in the support of $\hat{\mu}$. Most of the classical results give conditions to the rate of decay with some regularity properties or to the porosity of the support such that the support of the transform can not have any gaps. We focus fully on the results that give conditions to the support, but good examples of the classical results of the rate of decay case can be found by Levinson [33] and de Branges [9].

The main classical result we present is the Beurling gap theorem. It gives a metric for the porosity of the support of μ such that the support of $\hat{\mu}$ can not have any gaps. We will now give the definition of this metric:

Definition 4.6. Let $\{I_n\}_{n\in\mathbb{N}}$ be a family of disjoint intervals on the real line. This family of intervals is called *long* if

$$\sum_{n\in\mathbb{N}} \frac{|I_n|^2}{1+\operatorname{dist}^2(0,I_n)} = \infty,$$

and *short* if the sum is finite.

This metric makes an appearance with many uncertainty results, to name an example in the famous Beurling-Malliavin theory. The statement of the Beurling Gap Theorem can now be expressed simply:

Theorem 4.7 (Beurling Gap Theorem)**.** *Let μ be a finite complex measure on the real line and suppose the complement of the support of μ is long. If $\hat{\mu}$ vanishes on any interval in $\mathbb{R}$, then $\mu \equiv 0$.*

The proof of this significant result can be found in [3].

The next logical step from the classical results is to find a maximal size of gap in support of $\hat{\mu}$ for a measure μ with a given support. Formally, for a given closed set $X \subset \mathbb{R}$, the *gap characteristic* G_X is defined as

$$G_X = \sup\{a : \exists \mu \in M, \mu \not\equiv 0, \operatorname{supp}\mu \subset X, \hat{\mu} = 0 \text{ on } [0,a]\},$$

where M is the set of all finite Borel complex measures. It took considerable time after the classical results, before Poltoratski [40] gave a formula for the gap characteristic of a closed set X. The main condition of the formula is the existence of a so-called *d-uniform sequence* in the set X. These sequences are characterised by a sort of density and energy condition. For more details the reader should refer to the original article, or the excellent book by the same author [41], where he studies different problems of uncertainty and finds links to different classical problems such as the Pólya-Levinson problem.

Chapter 5

Density based uncertainty principle and Fourier quasicrystals

5.1 Fourier quasicrystals

Before we start, we give two definitions that are important to the discussion of quasicrystals.

Definition 5.1. A set $\Lambda \subset \mathbb{R}^d$ is called *uniformly discrete* if

$$\inf_{\lambda,\lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.$$

Definition 5.2. Let $v_1, v_2, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^d$. Then the set

$$L = \left\{ \sum_{i=1}^{n} x_i v_i : x_i \in \mathbb{Z} \right\}$$

is called the *lattice* generated by $v_1, v_2, \ldots, v_n$. The vectors $v_1, v_2, \ldots, v_n$ are called the *basis* of the lattice.

We call the number of vectors in the basis of a lattice the *rank* of the lattice. If the rank of the lattice L is equal to the dimension of the space $\mathbb{R}^d$, we say that L is a full rank lattice. With these concepts defined we can move on to the quasicrystals.

The definition of Fourier quasicrystals is not unambiguous, but a usual definition is that a complex discrete measure μ on $\mathbb{R}^d$ is a *Fourier quasicrystal* if its Fourier transform $\hat{\mu}$ is also discrete. If Λ is the support of quasicrystal μ and S support of $\hat{\mu}$ then

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \quad \text{and} \quad \hat{\mu} = \sum_{s \in S} b_s \delta_s, \tag{5.1}$$

where $a_\lambda \neq 0$ for $\lambda \in \Lambda$, $b_s \neq 0$ for $s \in S$ and δ_x is the Dirac delta mass on x.

The interest in quasicrystals started after the discovery by Shechtman in 1982 of a material with atomic structure without periodicity that has a diffraction pattern consisting solely of spots. This great discovery was initially almost universally rejected by the crystallography community, with even the Nobel award-winning Linus Pauling going as far as to state that "There is no such thing as quasicrystals, only quasi-scientists" according to Shechtman. In 2011 Shechtman was finally awarded the Nobel prize in chemistry for his discoveries.

The simplest example of a Fourier quasicrystals is the Dirac comb

$$\mu = \sum_{n \in \mathbb{Z}} \delta_n$$

since by the classical Poisson summation formula for any Schwarz function f

$$\langle \hat{\mu}, f \rangle = \langle \mu, \hat{f} \rangle = \sum_{n \in \mathbb{Z}} \hat{f}(n) = \sum_{n \in \mathbb{Z}} f(n) = \langle \mu, f \rangle,$$

and thus $\hat{\mu} = \mu$. Similarly any measure μ fulfilling (5.1) is linked to a Poisson summation formula of the form

$$\sum_{\lambda \in \Lambda} a_\lambda \hat{f}(\lambda) = \sum_{s \in S} b_s f(s). \tag{5.2}$$

Proving existence of Poisson summation formulas other than the ones over a lattice and its dual had been a known problem in harmonic analysis since the 50's when the results by Kahane-Mandelbrojt [24] and Guinand [14] were published. Kahane and Mandelbrojt showed that the existence of a Poisson summation formula of the form (5.2) in the case where Λ and S are locally finite is equivalent to the corresponding Dirichlet series fulfilling a certain functional equation. Guinand proved the existence of such Poisson formula for sets $\Lambda = S = \{\sqrt{k + 1/9} : k \in \mathbb{N}\}$.

Other important developments before 1980's were the discovery of Penrose tilings [38] in 1974 and the model sets by Meyer [34] in 1972, both constructions giving a Fourier quasicrystal. With all of this there was a lot of theory to work with for the mathematicians who got interested in the subject after the discovery of Shechtman.

We quickly explain the construction of a *cut-and-project set*, a special case of Meyer's model sets, as an example of a classical way of constructing quasicrystals. Let L be a full rank lattice in $\mathbb{R}^{n+m}$. We take a compact set $K \subset \mathbb{R}^m$ often called the mask or window, take all elements of the lattice L inside $\mathbb{R}^n \times K$ and then project them onto $\mathbb{R}^n$. The set we get is the cut-and-project set

$$\Lambda = \{\mathrm{pr}_{\mathbb{R}^n}(x) : x \in L, \mathrm{pr}_{\mathbb{R}^m}(x) \in K\}.$$

In the case of model sets we use the same method, but in $\mathbb{R}^n \times G$, where G is a locally compact Abelian group.

To this date quasicrystals remain an active field of study. There is great interest in classification of the set of Fourier crystals but since many different constructions of Fourier quasicrystals on increasingly nontrivial sets have been found this effort seems rather difficult. On the other hand the fruits of such effort could be substantial and even a link to the Riemann Hypothesis has been theorized by Dyson in his inspiring lecture [11]. During the 2010's considerable steps forward have been made. For example Lev and Olevskii found an answer to two questions that had remained open for a rather long time with the following theorems:

Theorem 5.3 (Lev-Olevskii [30][31])**.** *Let μ be a measure of the form* (5.1) *on $\mathbb{R}$ with Λ and S both uniformly discrete. Then there exists a lattice L such that μ is of the form*

$$\mu = \sum_{j=1}^{N} \sum_{\lambda \in \tau_j + L} P_j(\lambda)\delta_\lambda$$

where $\tau_j \in \mathbb{R}$ and P_j is a trigonometric polynomial for all $1 \leq j \leq N$. For $\mathbb{R}^d$ where $d > 1$, the same result holds if μ is also positive.

Theorem 5.4 (Lev-Olevskii [32])**.** *There exists a non-trivial measure μ of the form* (5.1) *on $\mathbb{R}$ for which Λ and S are both closed discrete sets and Λ contains at most finitely many elements of any arithmetic progression.*

The converse of Theorem 5.3 is also true, so we can build classification of quasicrystals in the one-dimensional case where Λ and S are uniformly discrete. On the other hand the Theorem 5.4 tells us that if we loosen the requirement of uniform discreteness, we find quasicrystals supported on a strongly non-periodic set.

The study of Fourier quasicrystals of the form

$$\mu = \sum_{\lambda \in \Lambda} \delta_\lambda \tag{5.3}$$

has also advanced greatly. In 2020 Kurasov and Sarnak [29] found an example of a Fourier quasicrystal of the form (5.3) where Λ is strongly aperiodic, with the help of so-called stable polynomials. This year Olevskii and Ulanovskii [37] went a step further and characterized all Fourier quasicrystals of unit mass.

Theorem 5.5 (Olevskii-Ulanovskii)**.** *Let Λ be a discrete set. Then a measure of the form* (5.3) *is a Fourier quasicrystal if and only if there exists an exponential polynomial*

$$p(t) = \sum_{i=1}^{N} c_i e^{2\pi i \eta_i t}, \quad N \in \mathbb{N}, c_i \in \mathbb{C}, \eta_i \in \mathbb{R}$$

with real simple zeros such that $\Lambda = \{z \in \mathbb{C} : p(z) = 0\}$.

Moving closer to the main result of this chapter we talk about Fourier interpolation. Let Σ be a class of well-behaved functions. The idea is to try to find two sequences Λ and Γ of real numbers without finite accumulation points and for these sequences the corresponding sequences of nice enough functions $(\Phi_\lambda)_{\lambda\in\Lambda}$ and $(\Psi_\gamma)_{\gamma\in\Gamma}$ such that we can represent every function $f \in \Sigma$ in the form

$$f(x) = \sum_{\lambda\in\Lambda} \Phi_\lambda(x) f(\lambda) + \sum_{\gamma\in\Gamma} \Psi_\gamma(x)\hat{f}(\gamma). \tag{5.4}$$

What we mean by "well-behaved" of course depends on the context, but the minimum requirements are such that both the series converge and the formula works predictably when Fourier transformed. Recently the existence of such interpolation formulas have been proved by for example Radchenko-Viazovska [42] and Bondarenko-Radchenko-Seip [7]. We notice that if f is a function with representation (5.4) such that $f(\lambda) = 0$ for all $\lambda \in \Lambda$ and $\hat{f}(\gamma) = 0$ for all $\gamma \in \Gamma$, we have $f = 0$. Therefore all such interpolation formulas give us an uncertainty principle on Σ. Also Fourier interpolation on a class of functions containing the Schwarz functions gives us a quasicrystal $\mu_x = \sum_{\gamma\in\Gamma} \Psi_\gamma(x)\delta_\gamma$ since by (5.4)

$$\langle \widehat{\mu_x}, f\rangle = \langle \mu_x, \hat{f}\rangle = \sum_{\gamma\in\Gamma} \Psi_\gamma(x)\hat{f}(\gamma) = f(x) - \sum_{\lambda\in\Lambda} \Phi_\lambda(x) f(\lambda) = \langle \delta_x - \sum_{\lambda\in\Lambda} \Phi_\lambda \delta_\lambda, f\rangle,$$

giving us $\widehat{\mu_x} = \delta_x - \sum_{\lambda\in\Lambda} \Phi_\lambda(x)\delta_\lambda$.

5.2 Sobolev space closed under Fourier transform

In the results of this chapter we will make use of the function space $\mathcal{H} = \{f \in H^1(\mathbb{R}) : \hat{f} \in H^1(\mathbb{R})\}$ with the norm induced by the inner product

$$\langle f, g\rangle_{\mathcal{H}} = \langle f, g\rangle_{H^1} + \langle \hat{f}, \hat{g}\rangle_{H^1}.$$

This means that the norm induced by the inner product can be given as

$$\|f\|_{\mathcal{H}} = (\langle f, f\rangle_{\mathcal{H}})^{\frac{1}{2}} = \left(\langle f, f\rangle_{H^1} + \langle \hat{f}, \hat{f}\rangle_{H^1}\right)^{\frac{1}{2}} = \left(\|f\|_{H^1}^2 + \|\hat{f}\|_{H^1}^2\right)^{\frac{1}{2}}.$$

It is straightforward to conclude that $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ really is an inner product. Next we recall that $\mathcal{F}\left(\hat{f}\right)(x) = f(-x)$ which immediately gives us that the norm of the Fourier transform of f is

$$\|\hat{f}\|_{\mathcal{H}} = \left(\|\hat{f}\|_{H^1}^2 + \|\mathcal{F}\left(\hat{f}\right)\|_{H^1}^2\right)^{\frac{1}{2}} = \left(\|\hat{f}\|_{H^1}^2 + \|f\|_{H^1}^2\right)^{\frac{1}{2}} = \|f\|_{\mathcal{H}}$$

making the Fourier transform an isometry from $\mathcal{H}$ to itself. Since $\mathcal{H}$ is closed under Fourier transform, for any $f \in \mathcal{H}$ we have $\mathcal{F}^3(f) \in \mathcal{H}$ and $\mathcal{F}(\mathcal{F}^3(f)) = \mathcal{F}^4(f) = f$, making $\mathcal{F}$ surjective. Clearly $\hat{f} = 0$ implies $f = 0$ and we can finally conclude that the linear operator $\mathcal{F}\colon \mathcal{H} \mapsto \mathcal{H}$ is a bijective isometry.

Later on we would like to utilize the properties of reproducing kernel Hilbert spaces.

Theorem 5.6. *The space $\mathcal{H}$ is a reproducing kernel Hilbert space.*

Proof. We start by showing that $\mathcal{H}$ is complete and therefore a Hilbert space. Let (f_n) be a Cauchy sequence in $\mathcal{H}$. By the definition of the norm, we know that both (f_n) and $(\hat{f}_n)$ are Cauchy sequences in H^1. Since H^1 is complete, there exists functions $f, g \in H^1$ such that $f_n \to f$ and $\hat{f}_n \to g$ when $n \to \infty$. By definition the convergence in H^1 implies convergence in L^2. Therefore by Plancherel

$$\|\hat{f}_n - \hat{f}\|_{L^2} = \|f_n - f\|_{L^2} \to 0$$

when $n \to \infty$ and therefore $\hat{f}_n \to \hat{f}$ in L^2. Hence $g = \hat{f}$. From the definition of the norm of $\mathcal{H}$ we know that since $f_n \to f$ and $\hat{f}_n \to \hat{f}$ in H^1 where $f, \hat{f} \in H^1$, we can say that the sequence (f_n) converges to $f \in \mathcal{H}$ making $\mathcal{H}$ complete.

Now we only need to show that the linear evaluation functional $L_x\colon f \mapsto f(x)$ is continuous for all $x \in \mathbb{R}$. This gives us for every $x \in \mathbb{R}$ the function $k_x \in \mathcal{H}$ with the reproducing property $f(x) = \langle f, k_x \rangle$ for all $f \in \mathcal{H}$ by Riesz represantation theorem and makes $\mathcal{H}$ a reproducing kernel Hilbert space. The continuity follows again from using Fourier inversion and Cauchy-Schwarz

$$\begin{aligned}
|L_x(f)| = |f(x)| &\le \int_{\mathbb{R}} |\hat{f}(\xi) e^{2\pi i x \xi}| \, \mathrm{d}\xi = \int_{\mathbb{R}} |\hat{f}(\xi)| (1 + 4\pi^2 \xi^2)^{\frac{1}{2}} (1 + 4\pi^2 \xi^2)^{-\frac{1}{2}} \, \mathrm{d}\xi \\
&\le \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + 4\pi^2 \xi^2) \, \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{1 + 4\pi^2 \xi^2} \, \mathrm{d}\xi \right)^{\frac{1}{2}} \lesssim \|f\|_{H^1} \le \|f\|_{\mathcal{H}}
\end{aligned}$$

which finishes the proof. □

As a final property of the space $\mathcal{H}$ we show that for every $\xi \in \mathbb{R}$ there exists a function $h_\xi \in \mathcal{H}$ with the reproducing property $\hat{f}(\xi) = \langle f, h_\xi \rangle$. This can be done utilizing the same strategy as before. Let $\xi \in \mathbb{R}$. We define the functional $F_\xi\colon f \mapsto \hat{f}(\xi)$. The continuity of F_ξ can be again shown with Cauchy-Schwarz. Let $f \in \mathcal{H}$. Now

$$\begin{aligned}
|F_\xi(f)| = |\hat{f}(\xi)| &\le \int_{\mathbb{R}} |f(x) e^{-2\pi i x \xi}| \, \mathrm{d}x = \int_{\mathbb{R}} |f(x)| (1 + 4\pi^2 x^2)^{\frac{1}{2}} (1 + 4\pi^2 x^2)^{-\frac{1}{2}} \, \mathrm{d}x \\
&\le \left(\int_{\mathbb{R}} |f(x)|^2 (1 + 4\pi^2 x^2) \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{1 + 4\pi^2 x^2} \, \mathrm{d}\xi \right)^{\frac{1}{2}} \lesssim \|\hat{f}\|_{H^1} \le \|f\|_{\mathcal{H}},
\end{aligned}$$

making F_ξ a continuous operator. Again by Riesz representation theorem we can conclude the existence of a function $h_\xi \in \mathcal{H}$ such that for every $f \in \mathcal{H}$ we have $\hat{f}(\xi) = F_\xi(f) = \langle f, h_\xi \rangle$.

5.3 A density based uncertainty principle by Kulikov-Nazarov-Sodin

For the rest of this chapter we examine the results in the upcoming article by Kulikov-Nazarov-Sodin [28]. The proofs are based on the sketches kindly provided by Sodin. We start with the uncertainty principle that gives a clear restriction to the simultaneous density of the sets where a function and its Fourier transform vanish.

Theorem 5.7. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ and $\Gamma = (\gamma_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ be strictly increasing sequences with no finite accumulation points for which there exists $k < 1$ such that the density conditions*

$$\sup_{n\in\mathbb{Z}} \max\{|\lambda_n|, |\lambda_{n+1}|\}(\lambda_{n+1} - \lambda_n) \leq \frac{1}{2}k \tag{5.5}$$

$$\sup_{n\in\mathbb{Z}} \max\{|\gamma_n|, |\gamma_{n+1}|\}(\gamma_{n+1} - \gamma_n) \leq \frac{1}{2}k \tag{5.6}$$

hold. If a function $f \in \mathcal{H}$ satisfies $f|_\Lambda = 0$ and $\hat{f}|_\Gamma = 0$, then $f = 0$.

The proof of this theorem is basically a nifty application of Wirtinger's inequality. Our proof of the inequality will be based on the one in [10].

Lemma 5.1 (Wirtinger inequality)**.** *Let (a, b) be a bounded interval, $f \in H^1(a, b)$ and $f(a) = f(b) = 0$. Then*

$$\int_a^b |f(x)|^2 \, \mathrm{d}x \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \, \mathrm{d}x.$$

Proof. We first prove the inequality for the interval $\left(0, \frac{1}{2}\right)$. Let $f \in H^1\left(0, \frac{1}{2}\right)$ be such that $f(0) = f\left(\frac{1}{2}\right)$. We extend f as an odd function to $\left(-\frac{1}{2}, \frac{1}{2}\right)$. By Theorem 2.34 we know that the extension is a function in $H^1\left(-\frac{1}{2}, \frac{1}{2}\right)$. Moreover, by assumption the extension has matching values at the endpoints and thus f is a periodic continuous function. We use L^2-theory of the Fourier series to finish the proof. Since f is odd, we have

$$\hat{f}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) \, \mathrm{d}x = 0.$$

Moreover, since $f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right)$ the function f has a periodic weak derivative in L^2 by Theorem 2.34. We use Plancherel's formula to say

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |f'(x)|^2 \,\mathrm{d}x = \sum_{n\in\mathbb{Z}} |\hat{f}'(n)|^2 = 4\pi^2 \sum_{n\neq 0} n^2 |\hat{f}(n)|^2 \geq 4\pi^2 \sum_{n\neq 0} |\hat{f}(n)|^2 = 4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 \,\mathrm{d}x.$$

By the oddness of f and the knowledge that the weak derivative of an odd function is even almost everywhere we can state the previous inequality as

$$\int_0^{\frac{1}{2}} |f(x)|^2 \,\mathrm{d}x \leq \left(\frac{1}{2}\right)^2 \frac{1}{\pi^2} \int_0^{\frac{1}{2}} |f'(x)|^2 \,\mathrm{d}x,$$

and the claim holds for the interval $\left(0, \frac{1}{2}\right)$.

For an arbitrary bounded interval (a, b) and $f \in H^1(a, b)$ we define the function $g(x) = f(2(b-a)x + a)$. Then $g \in H^1(0, \frac{1}{2})$ and $g(0) = g\left(\frac{1}{2}\right) = 0$ so by a change of variables

$$\begin{aligned}
\int_a^b |f(x)|^2 \,\mathrm{d}x &= 2(b-a) \int_0^{\frac{1}{2}} |f(2(b-a)x + a)|^2 \,\mathrm{d}x = 2(b-a) \int_0^{\frac{1}{2}} |g(x)|^2 \,\mathrm{d}x \\
&\leq \frac{2(b-a)}{4\pi^2} \int_0^{\frac{1}{2}} |g'(x)|^2 \,\mathrm{d}x = \frac{(b-a)}{2\pi^2} \int_0^{\frac{1}{2}} |f'(2(b-a)x + a)|^2 4(b-a)^2 \,\mathrm{d}x \\
&= \frac{(b-a)^2}{\pi^2} \int_0^{\frac{1}{2}} |f'(2(b-a)x + a)|^2 2(b-a) \,\mathrm{d}x = \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \,\mathrm{d}x,
\end{aligned}$$

so the claim holds for the interval (a, b). □

Now that we have the Wirtinger inequality, the proof of the uncertainty principle is relatively straightforward.

Proof of Theorem 5.7. Let the sequences Λ, Γ, and the function f be as described in the theorem. We first use the Wirtinger inequality to obtain

$$\begin{aligned}
\int_{\mathbb{R}} x^2 |f(x)|^2 \,\mathrm{d}x &= \sum_{n\in\mathbb{Z}} \int_{\lambda_n}^{\lambda_{n+1}} x^2 |f(x)|^2 \,\mathrm{d}x \leq \sum_{n\in\mathbb{Z}} \max\{|\lambda_n|^2, |\lambda_{n+1}|^2\} \int_{\lambda_n}^{\lambda_{n+1}} |f(x)|^2 \,\mathrm{d}x \\
&\leq \sum_{n\in\mathbb{Z}} \max\{|\lambda_n|^2, |\lambda_{n+1}|^2\} \frac{(\lambda_{n+1} - \lambda_n)^2}{\pi^2} \int_{\lambda_n}^{\lambda_{n+1}} |f'(x)|^2 \,\mathrm{d}x,
\end{aligned}$$

and then by using the density assumption of the sequence Λ and the Plancherel theorem we get

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \max\{|\lambda_n|^2, |\lambda_{n+1}|^2\} \frac{(\lambda_{n+1}-\lambda_n)^2}{\pi^2} \int_{\lambda_n}^{\lambda_{n+1}} |f'(x)|^2 \,\mathrm{d}x &\leq \frac{k^2}{(2\pi)^2} \sum_{n\in\mathbb{Z}} \int_{\lambda_n}^{\lambda_{n+1}} |f'(x)|^2 \,\mathrm{d}x \\
&= \frac{k^2}{(2\pi)^2} \int_{\mathbb{R}} |f'(x)|^2 \,\mathrm{d}x \\
&= \frac{k^2}{(2\pi)^2} \int_{\mathbb{R}} |\mathcal{F}(f')(\xi)|^2 \,\mathrm{d}\xi \\
&= k^2 \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \,\mathrm{d}\xi.
\end{aligned}$$

By denoting $g(\xi) = \hat{f}(\xi)$, we can use the previous estimate with the sequence Γ to say

$$\begin{aligned}
\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}} \xi^2 |g(\xi)|^2 \,\mathrm{d}\xi &\leq k^2 \int_{\mathbb{R}} x^2 |\hat{g}(x)|^2 \,\mathrm{d}x \\
&= k^2 \int_{\mathbb{R}} x^2 |f(-x)|^2 \,\mathrm{d}x = k^2 \int_{\mathbb{R}} x^2 |f(x)|^2 \,\mathrm{d}x,
\end{aligned}$$

and by combining these estimates we get that

$$\int_{\mathbb{R}} x^2 |f(x)|^2 \,\mathrm{d}x \leq k^4 \int_{\mathbb{R}} x^2 |f(x)|^2 \,\mathrm{d}x$$

where $k < 1$. This implies that the integral vanishes and thus $f = 0$. □

This result in itself is significant in the way that it gives a broad spectrum of pairs of discrete sets for which functions and their Fourier transforms cannot vanish, yet the scope of the article is much wider. The next result gives the uncertainty result a quantitative version, in the same way the result by Amrein-Berthier was a quantitative version of the result by Benedicks in chapter 4.

Theorem 5.8. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ and $\Gamma = (\gamma_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ be strictly increasing sequences with no finite accumulation points for which there exists $k < 1$ such that the density conditions (5.5) and (5.6) hold. If $f \in \mathcal{H}$, then*

$$\|f\|_{\mathcal{H}}^2 \lesssim \sum_{n\in\mathbb{Z}} (1+|\lambda_n|)|f(\lambda_n)|^2 + \sum_{n\in\mathbb{Z}} (1+|\gamma_n|)|\hat{f}(\gamma_n)|^2. \tag{5.7}$$

We again start by introducing handy lemmas. First, we prove a modification of Wirtinger inequality that enables us to use a similar idea as in the proof of the previous theorem.

Lemma 5.2 (Modified Wirtinger inequality). *Let (a, b) be a bounded interval and $f \in H^1(a, b)$. Then for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that*

$$\int_a^b |f(x)|^2 \, \mathrm{d}x \leq (1+\varepsilon)\frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 \, \mathrm{d}x + C_\varepsilon (b-a)(|f(a)|^2 + |f(b)|^2).$$

Proof. We again first prove the claim for the interval $(0, 1)$. Let $f \in H^1(0, 1)$ and $\varepsilon > 0$. To make use of the Wirtinger inequality we define the function $g(x) = f(x) - xf(1) - (1-x)f(0)$ and immediately note that $g \in H^1(0, 1)$ with $g(0) = g(1) = 0$. By using the Wirtinger inequality we get that

$$\int_0^1 |g(x)|^2 \, \mathrm{d}x \leq \frac{1}{\pi^2} \int_0^1 |g'(x)|^2 \, \mathrm{d}x = \frac{1}{\pi^2} \int_0^1 |f'(x) - f(1) + f(0)|^2 \, \mathrm{d}x,$$

and by applying the triangle inequality we get that

$$\begin{aligned} \|g\|_{L^2} &\leq \frac{1}{\pi}\left(\|f'\|_{L^2} + \left(\int_0^1 |f(1)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_0^1 |f(0)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \right) \\ &= \frac{1}{\pi}(\|f'\|_{L^2} + |f(1)| + |f(0)|). \end{aligned} \tag{5.8}$$

We also apply the reverse triangle inequality to get

$$\begin{aligned} \|g\|_{L^2} &\geq \|f\|_{L^2} - \left(\int_0^1 |xf(1)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} - \left(\int_0^1 |(1-x)f(0)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\ &= \|f\|_{L^2} - \frac{|f(1)|}{\sqrt{3}} - \frac{|f(0)|}{\sqrt{3}} \end{aligned} \tag{5.9}$$

and by combining (5.8) and (5.9) we can say that for some $C > 0$

$$\|f\|_{L^2}^2 \leq \left(\frac{1}{\pi}\|f'\|_{L^2} + C(|f(0)| + |f(1)|) \right)^2. \tag{5.10}$$

We modify the inequality $2xy \leq x^2 + y^2$ by assigning $x' = x\sqrt{\varepsilon}$ and $y' = \frac{y}{\sqrt{\varepsilon}}$ and with this we get that $(x+y)^2 = x^2 + y^2 + 2xy \leq (1+\varepsilon)x^2 + \left(1 + \frac{1}{\varepsilon}\right) y^2$. We use this in the right-hand side of (5.10) to get

$$
\begin{aligned}
\|f\|_{L^2}^2 &\le (1+\varepsilon)\frac{1}{\pi^2}\|f'\|_{L^2}^2 + \left(1+\frac{1}{\varepsilon}\right)C^2(|f(0)|+|f(1)|)^2 \\
&\le (1+\varepsilon)\frac{1}{\pi^2}\|f'\|_{L^2}^2 + \left(1+\frac{1}{\varepsilon}\right)2C^2(|f(0)|^2+|f(1)|^2)
\end{aligned}
$$

where the last step is due to $(a+b)^2 \le 2a^2+2b^2$. Therefore the claim holds for the interval $(0,1)$.

Consider next $f \in H^1(a,b)$ and $\varepsilon > 0$. We set $g(x) = f((b-a)x+a)$ and immediately note that $g \in H^1(0,1)$, $g(0) = f(a)$ and $g(1) = f(b)$. A simple change of variables yields

$$
\begin{aligned}
\int_a^b |f(x)|^2 \, \mathrm{d}x &= (b-a)\int_0^1 |f((b-a)x+a)|^2 \, \mathrm{d}x = (b-a)\int_0^1 |g(x)|^2 \, \mathrm{d}x \\
&\le (b-a)\left((1+\varepsilon)\frac{1}{\pi^2}\int_0^1 |g'(x)|^2 \, \mathrm{d}x + C_\varepsilon(|g(0)|^2+|g(1)|^2)\right) \\
&= (1+\varepsilon)\frac{(b-a)^2}{\pi^2}\int_0^1 |f'((b-a)x+a)|^2 (b-a) \, \mathrm{d}x + C_\varepsilon(b-a)(|f(a)|^2+|f(b)|^2) \\
&= (1+\varepsilon)\frac{(b-a)^2}{\pi^2}\int_a^b |f'(x)|^2 \, \mathrm{d}x + C_\varepsilon(b-a)(|f(a)|^2+|f(b)|^2),
\end{aligned}
$$

which proves the result for an arbitrary bounded interval (a,b). □

The second lemma is a consequence of Heisenberg's inequality.

Lemma 5.3. *Let $f \in \mathcal{H}$. Then*

$$
\|f\|_{L^2(\mathbb{R})}^2 \le 2\pi\left(\int_{\mathbb{R}} x^2|f(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right).
$$

Proof. By Theorem 2.41 we have

$$
\|f\|_{L^2(\mathbb{R})}^2 \le 4\pi\sqrt{\int_{\mathbb{R}} x^2|f(x)|^2 \, \mathrm{d}x}\sqrt{\int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2 \, \mathrm{d}\xi},
$$

and by simple application of Young's inequality we obtain

$$
4\pi\sqrt{\int_{\mathbb{R}} x^2|f(x)|^2 \, \mathrm{d}x}\sqrt{\int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2 \, \mathrm{d}\xi} \le 2\pi\left(\int_{\mathbb{R}} x^2|f(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right),
$$

which finishes the proof. □

With the help of these lemmas we are ready to tackle the proof of Theorem 5.8. The main idea is similar to that of the proof of Theorem 5.7, but there are some technical details we need to take into account.

Proof of Theorem 5.8. Let $f \in \mathcal{H}$ and Λ, Γ be sequences as described in the statement of the theorem. Since $k < 1$, we may choose an $\varepsilon > 0$ such that $k^2(1+\varepsilon) < 1$. Applying the same estimates as in the proof of Theorem 5.7, this time with the modified Wirtinger inequality, we get

$$
\begin{aligned}
\int_{\mathbb{R}} x^2|f(x)|^2 \,\mathrm{d}x &= \sum_{n\in\mathbb{Z}} \int_{\lambda_n}^{\lambda_{n+1}} x^2|f(x)|^2 \,\mathrm{d}x \leq \sum_{n\in\mathbb{Z}} \max\{|\lambda_n|^2, |\lambda_{n+1}|^2\} \int_{\lambda_n}^{\lambda_{n+1}} |f(x)|^2 \,\mathrm{d}x \\
&\leq \sum_{n\in\mathbb{Z}} \max\{|\lambda_n|^2, |\lambda_{n+1}|^2\}(1+\varepsilon)\frac{(\lambda_{n+1}-\lambda_n)^2}{\pi^2} \int_{\lambda_n}^{\lambda_{n+1}} |f'(x)|^2 \,\mathrm{d}x \\
&\quad + C_\varepsilon \sum_{n\in\mathbb{Z}} \max\{|\lambda_n|^2, |\lambda_{n+1}|^2\}(\lambda_{n+1}-\lambda_n)(|f(\lambda_n)|^2 + |f(\lambda_{n+1})|^2) \\
&\leq \sum_{n\in\mathbb{Z}} (1+\varepsilon)\frac{k^2}{4\pi^2} \int_{\lambda_n}^{\lambda_{n+1}} |f'(x)|^2 \,\mathrm{d}x + C_\varepsilon \sum_{n\in\mathbb{Z}} \frac{k}{2} \max\{|\lambda_n|, |\lambda_{n+1}|\}(|f(\lambda_n)|^2 + |f(\lambda_{n+1})|^2) \\
&\leq (1+\varepsilon)\frac{k^2}{4\pi^2} \int_{\mathbb{R}} |f'(x)|^2 \,\mathrm{d}x + C'_\varepsilon \sum_{n\in\mathbb{Z}} \max\{|\lambda_n|, |\lambda_{n+1}|\}(|f(\lambda_n)|^2 + |f(\lambda_{n+1})|^2) \\
&= (1+\varepsilon)k^2 \int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2 \,\mathrm{d}\xi + C'_\varepsilon \sum_{n\in\mathbb{Z}} \max\{|\lambda_n|, |\lambda_{n+1}|\}(|f(\lambda_n)|^2 + |f(\lambda_{n+1})|^2).
\end{aligned}
$$

Analogously to the proof of Theorem 5.7, we again denote $g(\xi) = \hat{f}(\xi)$ and apply the previous inequality to g, this time using the sequence Γ. This yields us

$$
\begin{aligned}
\int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}} \xi^2|g(\xi)|^2 \,\mathrm{d}\xi \\
&\leq (1+\varepsilon)k^2 \int_{\mathbb{R}} x^2|\hat{g}(x)|^2 \,\mathrm{d}x + C'_\varepsilon \sum_{n\in\mathbb{Z}} \max\{|\gamma_n|, |\gamma_{n+1}|\}(|g(\gamma_n)|^2 + |g(\gamma_{n+1})|^2) \\
&= (1+\varepsilon)k^2 \int_{\mathbb{R}} x^2|f(-x)|^2 \,\mathrm{d}x + C'_\varepsilon \sum_{n\in\mathbb{Z}} \max\{|\gamma_n|, |\gamma_{n+1}|\}(|\hat{f}(\gamma_n)|^2 + |\hat{f}(\gamma_{n+1})|^2) \\
&= (1+\varepsilon)k^2 \int_{\mathbb{R}} x^2|f(x)|^2 \,\mathrm{d}x + C'_\varepsilon \sum_{n\in\mathbb{Z}} \max\{|\gamma_n|, |\gamma_{n+1}|\}(|\hat{f}(\gamma_n)|^2 + |\hat{f}(\gamma_{n+1})|^2).
\end{aligned}
$$

By summing the previous inequalities we obtain

$$\int_{\mathbb{R}} x^2|f(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi \le C''_{\varepsilon,k}\sum_{n\in\mathbb{Z}} \max\{|\gamma_n|,|\gamma_{n+1}|\}(|\hat{f}(\gamma_n)|^2+|\hat{f}(\gamma_{n+1})|^2)$$
$$+\, C''_{\varepsilon,k}\sum_{n\in\mathbb{Z}} \max\{|\gamma_n|,|\gamma_{n+1}|\}(|\hat{f}(\gamma_n)|^2+|\hat{f}(\gamma_{n+1})|^2),$$

where $C''_{\varepsilon,k} = \frac{C'_\varepsilon}{1-(1+\varepsilon)k^2} > 0$. We use this and Lemma 5.3 to estimate the norm of f in $\mathcal{H}$:

$$\begin{aligned}
\|f\|^2_{\mathcal{H}} &= \int_{\mathbb{R}}|\hat{f}(\xi)|^2\,\mathrm{d}\xi + \int_{\mathbb{R}}\xi^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi + \int_{\mathbb{R}}|f(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}}x^2|f(x)|^2\,\mathrm{d}x\\
&= 2\int_{\mathbb{R}}|f(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}}\xi^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi + \int_{\mathbb{R}}x^2|f(x)|^2\,\mathrm{d}x\\
&\le (1+\pi)\int_{\mathbb{R}}\xi^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi + (1+\pi)\int_{\mathbb{R}}x^2|f(x)|^2\,\mathrm{d}x \qquad (5.11)\\
&\le C''_{\varepsilon,k}(1+\pi)\sum_{n\in\mathbb{Z}}\max\{|\gamma_n|,|\gamma_{n+1}|\}(|\hat{f}(\gamma_n)|^2+|\hat{f}(\gamma_{n+1})|^2)\\
&\quad + C''_{\varepsilon,k}(1+\pi)\sum_{n\in\mathbb{Z}}\max\{|\gamma_n|,|\gamma_{n+1}|\}(|\hat{f}(\gamma_n)|^2+|\hat{f}(\gamma_{n+1})|^2).
\end{aligned}$$

To finalize the proof, we need to show that the inequalities

$$\max\{|\lambda_n|,|\lambda_{n+1}|\} \le 1+\min\{|\lambda_n|,|\lambda_{n+1}|\}$$
$$\max\{|\gamma_n|,|\gamma_{n+1}|\} \le 1+\min\{|\gamma_n|,|\gamma_{n+1}|\}$$

hold for all $n\in\mathbb{N}$. We notice that $(\lambda_{n+1}-\lambda_n) > 1$ cannot hold for any $n\in\mathbb{N}$. If it did, the density condition (5.5) would imply $\max\{|\lambda_n|,|\lambda_{n+1}|\} \le \frac{1}{2}$. These two conditions cannot hold simultaneously, hence $(\lambda_{n+1}-\lambda_n)\le 1$ for all $n\in\mathbb{N}$. This immediately gives us $\max\{|\lambda_n|,|\lambda_{n+1}|\} \le 1+\min\{|\lambda_n|,|\lambda_{n+1}|\}$ for all $n\in\mathbb{N}$. Identical deduction also holds for the sequence Γ. We apply the previous inequalities to the inequality (5.11) and obtain

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &\lesssim \sum_{n\in\mathbb{Z}} \max\{|\lambda_n|, |\lambda_{n+1}|\}(|f(\lambda_n)|^2 + |f(\lambda_{n+1})|^2) \\
&\quad + \sum_{n\in\mathbb{Z}} \max\{|\gamma_n|, |\gamma_{n+1}|\}(|\hat{f}(\gamma_n)|^2 + |\hat{f}(\gamma_{n+1})|^2) \\
&\lesssim \sum_{n\in\mathbb{Z}} (1 + \min\{|\lambda_{n+1}|, |\lambda_n|\})(|f(\lambda_{n+1})|^2 + |f(\lambda_n)|^2) \\
&\quad + \sum_{n\in\mathbb{Z}} (1 + \min\{|\gamma_{n+1}|, |\gamma_n|\})(|\hat{f}(\gamma_{n+1})|^2 + |\hat{f}(\gamma_n)|^2) \\
&\leq \sum_{n\in\mathbb{Z}} ((1 + |\lambda_{n+1}|)|f(\lambda_{n+1})|^2 + (1 + |\lambda_n|)|f(\lambda_n)|^2) \\
&\quad + \sum_{n\in\mathbb{Z}} ((1 + |\gamma_{n+1}|)|\hat{f}(\gamma_{n+1})|^2 + (1 + |\gamma_n|)|\hat{f}(\gamma_n)|^2) \\
&\lesssim \sum_{n\in\mathbb{Z}} (1 + |\lambda_n|)|f(\lambda_n)|^2 + \sum_{n\in\mathbb{Z}} (1 + |\gamma_n|)|\hat{f}(\gamma_n)|^2,
\end{aligned}$$

which finishes the proof. □

5.4 Fourier interpolation formula by Kulikov-Nazarov-Sodin

We are finally ready to present the main result of this chapter, the Fourier interpolation formula by Kulikov-Nazarov-Sodin.

Theorem 5.9 (Kulikov-Nazarov-Sodin)**.** *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ and $\Gamma = (\gamma_n)_{n\in\mathbb{Z}}$ be as in the previous theorems. Additionally, suppose there exists $q, M > 0$ such that*

$$M|\lambda_n| \geq |n|^q \quad \textit{and} \quad M|\gamma_n| \geq |n|^q$$

for large enough $|n|$. Then there exists sequences of functions $(\Phi_\lambda)_{\lambda\in\Lambda}, (\Psi_\gamma)_{\gamma\in\Gamma} \subset \mathcal{H}$ such that for every $f \in \mathcal{S}$

$$f(x) = \sum_{\lambda\in\Lambda} f(\lambda)\Phi_\lambda(x) + \sum_{\gamma\in\Gamma} \hat{f}(\gamma)\Psi_\gamma(x),$$

for every $x \in \mathbb{R}$, and the series converges in $\mathcal{H}$.

Proof. We concluded in Section 5.2 that there exists sequences of functions $(k_x)_{x\in\mathbb{R}}$ and $(h_\xi)_{\xi\in\mathbb{R}}$ such that $f(x) = \langle f, k_x\rangle$ and $\hat{f}(\xi) = \langle f, h_\xi\rangle$ for all $f \in \mathcal{H}$. The norm of the functions in both of the sequences are also uniformly bounded since for all $x \in \mathbb{R}$ there exists a $C > 0$ such that

$$\|k_x\|_{\mathcal{H}}^2 = |\langle k_x, k_x\rangle| = |k_x(x)| = \int_{\mathbb{R}} |\hat{k}_x(\xi)e^{2\pi i x\xi}|\,\mathrm{d}\xi = \int_{\mathbb{R}} |\hat{k}_x(\xi)|(1+4\pi^2\xi^2)^{\frac{1}{2}}(1+4\pi^2\xi^2)^{-\frac{1}{2}}\,\mathrm{d}\xi$$
$$\leq \left(\int_{\mathbb{R}} |\hat{k}_x(\xi)|^2(1+4\pi^2\xi^2)\,\mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{1+4\pi^2\xi^2}\,\mathrm{d}\xi\right)^{\frac{1}{2}} \leq C\|k_x\|_{H^1} \leq C\|k_x\|_{\mathcal{H}}.$$

Symmetrically, we also obtain $\|h_\xi\|_{\mathcal{H}}^2 \leq C\|h_\xi\|_{\mathcal{H}}$, and conclude that $\|k_x\|_{\mathcal{H}} \leq C$ and $\|h_\xi\|_{\mathcal{H}} \leq C$.

By Theorem 5.8 there exists sequences of coefficients $(a_\lambda)_{\lambda\in\Lambda}$ and $(b_\gamma)_{\gamma\in\Gamma}$ such that

$$\|f\|_{\mathcal{H}}^2 \leq \sum_{\lambda\in\Lambda} a_\lambda |f(\lambda)|^2 + \sum_{\gamma\in\Gamma} b_\gamma |\hat{f}(\gamma)|^2 \tag{5.12}$$

where $0 \leq a_\lambda = \mathcal{O}(1+|\lambda|)$ and $0 \leq b_\gamma = \mathcal{O}(1+|\gamma|)$. We define a linear operator X by setting

$$Xf = \sum_{\lambda\in\Lambda} a_\lambda f(\lambda)k_\lambda + \sum_{\gamma\in\Gamma} b_\gamma \hat{f}(\gamma)h_\gamma$$

and immediately note that by inequality (5.12) and an application of Cauchy-Schwarz

$$\|f\|_{\mathcal{H}}^2 \leq \sum_{\lambda\in\Lambda} a_\lambda |f(\lambda)|^2 + \sum_{\gamma\in\Gamma} b_\gamma |\hat{f}(\gamma)|^2 = \sum_{\lambda\in\Lambda} a_\lambda f(\lambda)\overline{f(\lambda)} + \sum_{\gamma\in\Gamma} b_\gamma \hat{f}(\gamma)\overline{\hat{f}(\gamma)}$$
$$= \sum_{\lambda\in\Lambda} a_\lambda f(\lambda)\langle k_\lambda, f\rangle + \sum_{\gamma\in\Gamma} b_\gamma \hat{f}(\gamma)\langle h_\gamma, f\rangle = \left\langle \sum_{\lambda\in\Lambda} a_\lambda f(\lambda)k_\lambda, f\right\rangle + \left\langle \sum_{\gamma\in\Gamma} b_\gamma \hat{f}(\gamma)h_\gamma, f\right\rangle$$
$$= \left\langle \sum_{\lambda\in\Lambda} a_\lambda f(\lambda)k_\lambda + \sum_{\gamma\in\Gamma} b_\gamma \hat{f}(\gamma)h_\gamma, f\right\rangle = \langle Xf, f\rangle \leq \|Xf\|_{\mathcal{H}}\|f\|_{\mathcal{H}},$$

from which we get that $\|f\|_{\mathcal{H}} \leq \|Xf\|_{\mathcal{H}}$, showing that X is an injective operator.

Next we want to show that $Xf \in \mathcal{H}$ for all $f \in \mathcal{S}$. Let $f \in \mathcal{S}$. By the uniform boundedness of the sequences $(k_x)_{x\in\mathbb{R}}$ and $(h_\xi)_{\xi\in\mathbb{R}}$, combined with the knowledge of the growth rate of the sequences $(a_\lambda)_{\lambda\in\Lambda}$ and $(b_\gamma)_{\gamma\in\Gamma}$, we have

$$\|Xf\|_{\mathcal{H}} \leq \Big\|\sum_{\lambda\in\Lambda} a_\lambda f(\lambda)k_\lambda\Big\|_{\mathcal{H}} + \Big\|\sum_{\gamma\in\Gamma} b_\gamma \hat{f}(\gamma)h_\gamma\Big\|_{\mathcal{H}} \leq \sum_{\lambda\in\Lambda} |a_\lambda f(\lambda)|\|k_\lambda\|_{\mathcal{H}} + \sum_{\gamma\in\Gamma} |b_\gamma \hat{f}(\gamma)|\|h_\gamma\|_{\mathcal{H}}$$
$$\lesssim \sum_{\lambda\in\Lambda} |a_\lambda f(\lambda)| + \sum_{\gamma\in\Gamma} |b_\gamma \hat{f}(\gamma)| \lesssim \sum_{\lambda\in\Lambda} (1+|\lambda|)|f(\lambda)| + \sum_{\gamma\in\Gamma} (1+|\gamma|)|\hat{f}(\gamma)|.$$

Since f is a Schwartz function, it decays faster than any polynomial, so

$$\sum_{\lambda\in\Lambda}(1+|\lambda|)|f(\lambda)| + \sum_{\gamma\in\Gamma}(1+|\gamma|)|\hat{f}(\gamma)| \lesssim \sum_{\lambda\in\Lambda}\left(\frac{1}{1+|\lambda|}\right)^{k-1} + \sum_{\gamma\in\Gamma}\left(\frac{1}{1+|\gamma|}\right)^{k-1}$$

for any $k \in \mathbb{N}$. We choose a $k \in \mathbb{N}$ such that $k \geq \frac{2+q}{q}$ and by the lower bound of the density of the sequences Λ and Γ

$$\begin{aligned}\sum_{\lambda\in\Lambda}\left(\frac{1}{1+|\lambda|}\right)^{k-1} + \sum_{\gamma\in\Gamma}\left(\frac{1}{1+|\gamma|}\right)^{k-1} &\lesssim \sum_{n\in\mathbb{Z}}\left(\frac{1}{1+|n|^q}\right)^{k-1} + \sum_{n\in\mathbb{Z}}\left(\frac{1}{1+|n|^q}\right)^{k-1} \\ &\leq \sum_{n\in\mathbb{Z}}\frac{1}{1+|n|^{q(k-1)}} + \sum_{n\in\mathbb{Z}}\frac{1}{1+|n|^{q(k-1)}} \lesssim \sum_{n\in\mathbb{Z}}\frac{1}{1+|n|^2} < \infty\end{aligned}$$

and finally we may conclude that $Xf \in \mathcal{H}$ for all $f \in \mathcal{S}$. This means that X maps the function space $\mathcal{S}$ bijectively to $V = X\mathcal{S}$ which is a subspace of $\mathcal{H}$. Since also $\|f\|_{\mathcal{H}} \leq \|Xf\|_{\mathcal{H}}$, we know that X has a linear inverse X^{-1} in V, which is continuous with respect to the norm in $\mathcal{H}$.

We want to extend the inverse operator X^{-1} to the whole of $\mathcal{H}$ while still preserving continuity with respect to the norm in $\mathcal{H}$. First of all by Theorem 2.23 we can continuously extend X^{-1} to $\overline{V}$. We denote this extension by X_{ex}^{-1}. Then we get the complete extension by defining $X_{\mathcal{H}}^{-1} = X_{ex}^{-1} \circ \mathrm{pr}_{\overline{V}}$. Clearly $X_{\mathcal{H}}^{-1}f = X^{-1}f$ for all $f \in V$. The operator $X_{\mathcal{H}}^{-1}$ is also continuous as a composition of two continuous operators.

We apply the extended inverse $X_{\mathcal{H}}^{-1}$ to the definition of the operator X and get

$$X_{\mathcal{H}}^{-1}Xf(x) = X_{\mathcal{H}}^{-1}\left(\sum_{\lambda\in\Lambda} f(\lambda)a_\lambda k_\lambda\right)(x) + X_{\mathcal{H}}^{-1}\left(\sum_{\gamma\in\Gamma}\hat{f}(\gamma)b_\gamma h_\gamma\right)(x),$$

which by the continuity of $X_{\mathcal{H}}^{-1}$ is equivalent to

$$f(x) = \sum_{\lambda\in\Lambda} f(\lambda)a_\lambda X_{\mathcal{H}}^{-1}k_\lambda(x) + \sum_{\gamma\in\Gamma}\hat{f}(\gamma)b_\gamma X_{\mathcal{H}}^{-1}h_\gamma(x).$$

We have our desired formula if we denote $\Phi_\lambda = a_\lambda X_{\mathcal{H}}^{-1}k_\lambda$ and $\Psi_\gamma = b_\gamma X_{\mathcal{H}}^{-1}h_\gamma$. □

We finally present a few examples to demonstrate the result. First of all we define the sequences $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ and $\Gamma = (\gamma_n)_{n\in\mathbb{Z}}$ by setting for all $n \in \mathbb{Z}$

$$\lambda_n = C_1 \operatorname{sgn}(n)n^\alpha \qquad \text{and} \qquad \gamma_n = C_2 \operatorname{sgn}(n)n^\beta,$$

where $0 < \alpha, \beta \leq \frac{1}{2}$ and $0 < C_1, C_2 < \frac{1}{\sqrt{2}}$. It is straightforward to check that the density conditions (5.5) and (5.6) hold for the sequences. The Theorem 5.9 tells us that the sequences Λ and Γ generate a Fourier interpolation formula of the form

$$f(x) = \sum_{\lambda \in \Lambda} f(\lambda)\Phi_\lambda(x) + \sum_{\gamma \in \Gamma} \hat{f}(\gamma)\Psi_\gamma(x),$$

for the Schwartz space of functions. For the second example we go a step further. The sequences

$$\lambda_n = C_1 \operatorname{sgn}(n) n^\alpha + C_1' \frac{1}{n} \sin(n) \qquad \text{and} \qquad \gamma_n = C_2 \operatorname{sgn}(n) n^\beta + C_2' \frac{1}{n} \sin(n),$$

with $0 < \alpha, \beta \leq \frac{1}{2}$ and suitable constants $C_1, C_1', C_2, C_2' > 0$ also generate a Fourier interpolation formula according to Kulikov-Nazarov-Sodin. Already the wide range of constants we may take in the first example show how flexible the result is, but the second example with the perturbation by the weighted sine term should make the robustness of Kulikov-Nazarov-Sodin even more evident.

The breakthrough result by Radchenko-Viazovska gave an interpolation formula for even Schwartz functions on sequences $\Lambda = \Gamma = \{\pm\sqrt{n} : n \in \mathbb{N}\}$. Bondarenko-Radchenko-Seip gave several interesting interpolation formulas, one of the most interesting being the formula for the sequences

$$\Lambda = \left\{ \frac{1}{4\pi} \log n : n \in \mathbb{N} \right\} \text{ and } \Gamma = \left\{ \frac{\rho - 1/2}{i} : \rho \text{ is a zero of the Riemann-zeta function} \right\},$$

where Riemann Hypothesis is assumed. These results, albeit very important, are very rigid compared to the one by Kulikov-Nazarov-Sodin, mostly because the properties of modular forms used in their proofs do not allow any sort of perturbation on the generating sets. The sequences used by Bondarenko-Radchenko-Seip and Radchenko-Viazovska are also non-redundant, meaning that no point can be removed from the sequences without the result failing. This is in stark contrast to how we saw Kulikov-Nazarov-Sodin working in our two examples. Thus Kulikov-Nazarov-Sodin offers a much greater variety of interpolation formulas, and consequently also a greater variety of Fourier quasicrystals and uncertainty principles.

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